Name: Solutions **Student ID:** ____________

S13 PHY321: Midterm 2

March 15, 2013

NOTE: Show all your work to maximize partial credit. *No credit* for unsupported answers.

Turn the front page only when advised by the instructor!

Check that your exam has all 4 problems. Total points: **25**

Formulas pertaining to the material:

Rocket equation: $v = -gt + u \log(m_0/m)$

Damped harmonic oscillator equation: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$

General solution: $x(t) = \mathrm{e}^{-\beta t}\left[A_1 \exp\left(\sqrt{\beta^2 - \omega_0^2}t\right) + A_2 \exp\left(-\sqrt{\beta^2 - \omega_0^2}t\right)\right]$

Driven harmonic oscillator equation: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos\omega t$

Amplitude of stationary driven oscillations: $D = \frac{f}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}}$

Phase lag of driven oscillations: $\delta = \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right)$

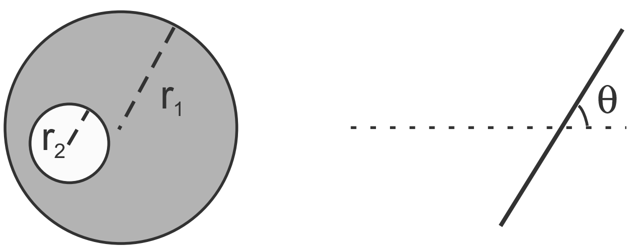
Avg. power in driven oscillator: $0 = \langle P_{\mathrm{drive}}\rangle + \langle P_{\mathrm{damp}}\rangle$

1. A circular plate of radius $r_1 = 30\,\text{cm}$ has a circular hole of radius $r_2 = 10\,\text{cm}$. The plate serves as a target for different projectile particles moving along the z-axis.

   (a) [2 pts] What is the cross section for the target plate getting hit by a point projectile particle, when the plate is perpendicular to the z-axis?

   (b) [2 pts] What is the cross section for the plate getting hit by a point projectile particle, when the plate is inclined at an angle of $\theta = 60°$ to the z-axis, rather than perpendicular?

   (c) [2 pts] What is the cross section for the target plate getting hit by a *spherical* projectile particle, of radius $a = 15\,\text{cm}$, while the plate is perpendicular to the z-axis?



a.) $\sigma = \pi(r_1^2 - r_2^2) = \pi(30^2 - 10^2) = 2513\ \text{cm}^2$

b.) $\sigma = \sigma_{(a)} \times \cos\theta$

c.) Since $a = 15\,\text{cm} > r_2$, hole doesn't matter

$\therefore \sigma = \pi(r_1 + a)^2$

(Over)

2. A rocket of initial mass $m_0 = 1.50 \times 10^4$ kg, equipped with an engine for which the speed of exhaust gas is $u = 3100$ m/s, is prepared for a vertical take-off from the ground.

   (a) [2 pts] What minimal thrust does the engine need to develop for the rocket to take off from the ground right away?

   (b) [2 pts] At what rate would the fuel burn then?

   (c) [3 pts] How long would it take for the rocket to reach the mass ratio of $m_0/m = 4.2$, under those circumstances?

a) $T \geq m_0 g$

b) $T = \alpha u \Rightarrow \alpha = \left|\frac{dm}{dt}\right| = \frac{T}{u}$

c) $\frac{dm}{dt} = -\alpha \Rightarrow m(t) - m_0 = -\alpha t$

$$t = \frac{m_0 - m(t)}{\alpha} = \frac{m_0}{\alpha}\left(1 - \overset{1/4.2}{\frac{m(t)}{m_0}}\right)$$

3. A particle of mass m is moving along the x-axis under the influence of a conservative force for which the potential energy is given by

$$U(x) = B + C\,x + R\,x^2\,,$$

where B, C and R are constants.

(a) [1 pt] Determine the equilibrium point x_0 for the particle, in terms of B, C, R and m.

(b) [1 pt] What conditions, if any, must be satisfied by the constants to make x_0 a stable equilibrium point?

(c) [3 pts] Find the period τ_0 of oscillations of the particle about the stable x_0, in terms of B, C, R and m.

a.) $\left.\frac{dU}{dx}\right|_{x_0} = 0 = C + 2Rx_0 \Rightarrow \boxed{x_0 = -\frac{C}{2R}}$

b.) $\left.\frac{d^2U}{dx^2}\right|_{x_0} > 0 \Rightarrow 2R > 0 \Rightarrow \boxed{R > 0}$

c) Near x_0, we Taylor expand $U(x) \simeq \underbrace{U(x_0)}_{\text{irrelevant constant. Ignore it.}} + (x-x_0)\underbrace{\left.\frac{dU}{dx}\right|_{x_0}}_{\to 0 \text{ from part a}} + \frac{1}{2}(x-x_0)^2\left.\frac{d^2U}{dx^2}\right|_{x_0}$

$\Rightarrow U(x) \simeq \frac{1}{2}(x-x_0)^2 U''(x_0)$

$\Rightarrow F = -\frac{dU}{dx} \simeq -(x-x_0)U''(x_0)$ compare to $F = -k(x-x_0)$ (Hooke's Law)

$\Rightarrow k = U''(x_0) = 2R$

and $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2R}{m}}$

$\boxed{\tau_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{2R}}}$

4. A particle of mass m, attached to a spring of spring constant k and subject to a damping force, is driven in one dimension by a force depending sinusoidally on time. The equation of motion of the particle is:

$$m\,\ddot{x} + b\,\dot{x} + k\,x = F_0\,\cos(\omega t)\,,$$

where m, b, k, F_0 and ω are constants.

(a) [2 pts] In a steady-state motion, the amplitude for the displacement of the particle is D. What is the amplitude for the velocity of the particle then, in terms of D and the constants above?

(b) [3 pts] What is the average rate at which the damping force carries out work on the particle, when the driving frequency is ω and the amplitude of steady-state motion is D, again in terms of D and the constants above?

(c) [2 pts] What is the average rate at which the driving force carries out work on the particle then?

a) $x(t) = D(\omega)\cos(\omega t - \delta)$

$\dot{x}(t) = -\omega D(\omega)\sin(\omega t - \delta)$ $\Rightarrow$ $\boxed{D_v(\omega) = \omega D(\omega)}$ where $D(\omega)$ on formula sheet

b) $dW = F_{damp}\,dx \Rightarrow \dfrac{dW}{dt} = F_{damp}\,\dot{x} = -b\,\dot{x}^2$

Now $\left\langle \dfrac{dW}{dt} \right\rangle = -b\left\langle \dot{x}^2 \right\rangle = -b\,D_v^2(\omega)\,\underbrace{\left\langle \sin^2(\omega t - \delta) \right\rangle}_{1/2}$

$$\Rightarrow \boxed{\left\langle \frac{dW}{dt} \right\rangle_{damp} = -\frac{b\,D^2(\omega)\,\omega^2}{2}}$$

c) $\left\langle \dfrac{dW}{dt} \right\rangle_{damp} + \left\langle \dfrac{dW}{dt} \right\rangle_{drive} = 0 \Rightarrow \left\langle \dfrac{dW}{dt} \right\rangle_{drive} = \dfrac{1}{2}\,b\,D^2(\omega)\,\omega^2$