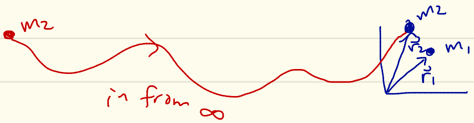
* 1 Last loose end from Ch5 - Potential energy in Gravitational Field

Question: How much work must we do to assemble N masses $m_1, \ldots m_N$ by bringing them in 1-by-1 from ∞ to their final positions $(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N)$

$W_1 = 0$ (no field to work against since all other masses @ ∞)

* Now bring m_2 in from ∞. Must work against the grav. field of m_1



$$\phi_1(\vec{r}) = \frac{-m_1}{|\vec{r}-\vec{r}_1|} G$$

$$W_2 = m_2 \phi_1(\vec{r}_2) - m_2 \cancelto{0}{\phi_1(\infty)} = - \frac{m_2 m_1}{|\vec{r}_2 - \vec{r}_1|} G$$

* Now bring in m_3 from ∞. Must work against grav. field of $m_1 + m_2$

$$W_3 = m_3 \left(\phi_1(\vec{r}_3) + \phi_2(\vec{r}_3) \right) - m_3 \left(\cancelto{0}{\phi_1(\infty)} + \cancelto{0}{\phi_2(\infty)} \right)$$

$$= -G \frac{m_3 m_1}{|\vec{r}_3 - \vec{r}_1|} - G \frac{m_3 m_2}{|\vec{r}_3 - \vec{r}_2|}$$

Generalizing, we see the total work to assemble N masses from ∞ to $(\vec{r}_1, \dots \vec{r}_N)$ is

$$W = W_1 + W_2 + \dots + W_N = \sum_{\substack{i,j \\ (i<j)}} \frac{-Gm_i m_j}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \frac{-Gm_i m_j}{|\vec{r}_i - \vec{r}_j|}$$

= total PE

Taking continuum limit as earlier

$$\boxed{W = U = -\frac{G}{2} \iint d^3\vec{r}\, d^3\vec{r}\,' \frac{\rho(\vec{r})\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} = \frac{1}{2} \int d^3 r\, \rho(\vec{r})\, \phi(\vec{r})}$$

$$\left(* \text{ used } \quad \phi(\vec{r}) = -G \int d^3\vec{r}\,' \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \right)$$

Alternative way to write entirely in terms of $\vec{g}$

$$\left.\begin{array}{l} \vec{\nabla} \cdot \vec{g} = -4\pi G \rho \\ \quad \shortparallel \\ -\nabla^2 \phi \end{array}\right\} \Rightarrow \boxed{\rho = \frac{1}{4\pi G} \nabla^2 \phi}$$

$$\Rightarrow \frac{1}{2} \int \rho \phi \, d^3 r = \frac{1}{8\pi G} \int \phi \nabla^2 \phi \, d^3 r$$

* but $\vec{\nabla}\cdot(\phi\vec{\nabla}\phi) = \phi\vec{\nabla}^2\phi + \vec{\nabla}\phi\cdot\vec{\nabla}\phi$

$$\Rightarrow \phi\nabla^2\phi = \vec{\nabla}\cdot(\phi\vec{\nabla}\phi) - |\vec{\nabla}\phi|^2$$

$$\Rightarrow U = \frac{1}{8\pi G}\left[\underbrace{\int \vec{\nabla}\cdot(\phi\vec{\nabla}\phi)\,d^3r}_{} - \int |\vec{g}|^2\,d^3r\right]$$

$\parallel$ Div. thm.

$$\oint_S \cancel{(\phi\vec{\nabla}\phi)}\cdot d\vec{A}$$

0 for S @ ∞.

End result: $\boxed{U = \frac{1}{2}\int \rho(\vec{r})\phi(\vec{r})\,d^3\vec{r} = -\frac{1}{8\pi G}\int d^3r\,|\vec{g}|^2}$

Example: (Prob. 5.14) Energy of a Uniform sphere $\rho = \frac{M}{\frac{4}{3}\pi R^3}$

$$4\pi r^2 g(r) = -4\pi G M_{enc} \Rightarrow g(r) = -\frac{GM_{enc}}{r^2}$$

$\Downarrow$

$$\vec{g}(r>R) = -\frac{GM}{r^2}\hat{r}$$

$$\vec{g}(r<R) = -\frac{GM_{enc}(r)}{r^2}\hat{r} \qquad M_{enc}(r) = \rho\,\tfrac{4}{3}\pi r^3 = M\frac{r^3}{R^3}$$

$$= -\frac{GMr}{R^3}\hat{r}$$

$$\Rightarrow U = -\frac{1}{8\pi G} 4\pi \int g^2 r^2 dr$$

$$= -\frac{1}{2G}\left[\int_0^R g^2 r^2 dr + \int_R^\infty g^2 r^2 dr\right]$$ ← Plug in g for each region

$$= -\frac{1}{2G} G^2 M^2 \left[\frac{1}{R^6}\int_0^R r^4 dr + \int_R^\infty \frac{1}{r^2} dr\right]$$

$$= -\frac{GM^2}{2}\left[\frac{1}{5R} + \frac{1}{R}\right]$$

$$= -\frac{3}{5}\frac{GM^2}{R}$$

<u>Exercise</u>: try to recover this using $U = \frac{1}{2}\int \rho(\vec{r})\phi(\vec{r})\, d^3r$

New topic: Central Force Motion, Ch 8

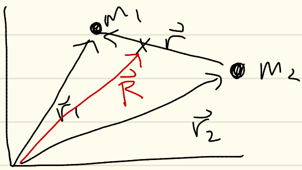
- celestial mechanics (planets, moons, etc.)
- atomic + molecular systems
- α decays in nuclei

} when described in QM

Reminder - Reduction of 2-body problem to effective 1-body

* recall relative/CM coordinates

$$\boxed{\begin{aligned} \vec{R} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \\ \vec{r} &= \vec{r}_1 - \vec{r}_2 \end{aligned}}$$



* Solve for $\vec{r}_1$ + $\vec{r}_2$ in terms of $\vec{R}, \vec{r}$

$$\boxed{\begin{aligned} \vec{r}_1 &= \vec{R} + \frac{m_2}{M}\vec{r} \\ \vec{r}_2 &= \vec{R} - \frac{m_1}{M}\vec{r} \end{aligned} \qquad (M = m_1+m_2)}$$

* let $\vec{F}_{12} = \vec{F}(\vec{r}) = -\vec{F}_{21}$

$$\left.\begin{aligned} \Rightarrow m_1\ddot{\vec{r}}_1 &= \vec{F}_{12}(\vec{r}) = \vec{F}(\vec{r}) \\ m_2\ddot{\vec{r}}_2 &= \vec{F}_{21}(\vec{r}) = -\vec{F}(\vec{r}) \end{aligned}\right\} \text{ express in } \vec{r} + \vec{R}$$

$$\left.\begin{aligned} m_1\ddot{\vec{R}} + \frac{m_1 m_2}{M}\ddot{\vec{r}} &= \vec{F}(\vec{r}) \\ m_2\ddot{\vec{R}} - \frac{m_2 m_1}{M}\ddot{\vec{r}} &= -\vec{F}(\vec{r}) \end{aligned}\right\} \text{ adding gives } (m_1+m_2)\ddot{\vec{R}} = M\ddot{\vec{R}} = 0$$

(i.e., COM moves like free particle)

* Using $\ddot{\vec{R}} = 0$ gives $\quad \frac{m_1 m_2}{M}\ddot{\vec{r}} = \vec{F}(\vec{r})$

Recall reduced mass $\quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{M}{m_1 m_2}$

=> We therefore have for the 2-particle system interacting via $\vec{F}_{12} = -\vec{F}_{21} \equiv \vec{F}(\vec{r})$

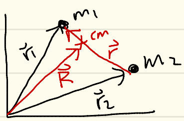
$$\boxed{\begin{aligned} M\ddot{\vec{R}} &= 0 \\ \mu\ddot{\vec{r}} &= \vec{F}(\vec{r}) \end{aligned}}$$

* Also, recall that for KE we can write

$$\left.\begin{aligned} T &= \tfrac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \tfrac{1}{2}m_2\dot{\vec{r}}_2^{\,2} \\ &\overset{\text{plugging in}}{\underset{\vec{r}_1(\vec{r},\vec{R}),\ \vec{r}_2(\vec{r},\vec{R})}{=}} \tfrac{1}{2}M\dot{\vec{R}}^{2} + \tfrac{1}{2}\mu\dot{\vec{r}}^{\,2} \end{aligned}\right\} \Rightarrow E = T + U = \underbrace{T_R}_{\text{cm coord}} + \underbrace{T_r}_{\text{relative coord}} + U(\vec{r})$$

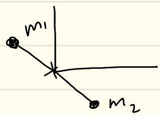
* Since the motion of $\vec{R}$ is trivial, we can ignore it + focus on motion in the relative coordinate $\vec{r}$.

* Let us go the the so-called COM frame to make this explicit



Lab frame

$$\xrightarrow[\text{frame}]{\text{COM}}$$



$$\vec{R} = 0$$
$$\vec{r}_1 = \frac{m_2}{M}\vec{r}$$
$$\vec{r}_2 = \frac{-m_1}{M}\vec{r}$$

Angular Momentum

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$

$*$ Recall, $\vec{P}_{tot} = m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2$

$$= m_1 \dot{\vec{R}} + \cancel{\frac{m_1 m_2}{M}} \dot{\vec{r}} + m_2 \dot{\vec{R}} - \cancel{\frac{m_2 m_1}{M}} \dot{\vec{r}}$$

$*$ In COM frame, $\vec{R} = 0 \; (\therefore \dot{\vec{R}} = 0) \Rightarrow \vec{P}_{tot} = 0$

$$\vec{p}_1 = -\vec{p}_2 \equiv \vec{p} = \mu \dot{\vec{r}}$$

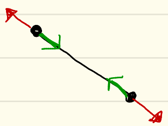
$$= \vec{r}_1 \times \vec{p} - \vec{r}_2 \times \vec{p}$$

$$= (\vec{r}_1 - \vec{r}_2) \times \vec{p}$$

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \cancel{\dot{\vec{r}} \times \vec{p}}^{\,0} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times (\mu \ddot{\vec{r}}) \stackrel{\text{EOM}}{=} \vec{r} \times \vec{F}$$

Now, for central forces, we have $\vec{F}(\vec{r}) = \hat{r} F(r)$



i.e., central forces obey so-called strong version of Newton's 2nd law

i) $\vec{F}_{12} = -\vec{F}_{21} \equiv \vec{F}$

(ii) $\vec{F}$ directed along $\vec{r}$.

$\Rightarrow$ For central forces $\quad \vec{r} \times \vec{F} = 0$

$$\Rightarrow \frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} = \underline{\text{constant}}.$$

* $\vec{L} = \vec{r} \times \vec{p}$ $(\perp \vec{L})$ at all times. Since $\vec{L}$ unchanging, that means $\vec{r}$ + $\vec{p}$ move in fixed plane $\perp$ to $\vec{L}$.

$\Downarrow$

I.e., we've reduced a 2-body problem in 3D to an effective 1-body problem in 2D! Use Polar coords to describe motion

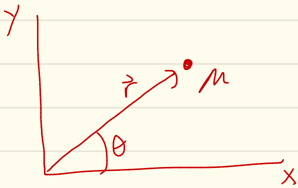
$x = r\cos\theta$

$y = r\sin\theta$

$\Downarrow$

$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$

$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$



$$\Rightarrow |\dot{\vec{r}}|^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$\boxed{T = \frac{1}{2}\mu|\dot{\vec{r}}|^2 = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right)}$$

* Rather than directly attacking the EOM $\mu\ddot{\vec{r}} = \hat{r}F(r)$, here we use a trick that $E = T + U$ & $\vec{L} = \vec{r} \times \vec{p}$ are constants of motion.

Angular momentum: $\dot{\vec{r}} = v_r\hat{r} + v_\theta\hat{\theta}$

$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

radial vel

angular vel.

$$\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times(\mu\dot{\vec{r}}) = \mu\vec{r}\times(\cancelto{0}{\dot{r}\hat{r}} + r\dot{\theta}\hat{\theta})$$

$$\Rightarrow \vec{L} = \mu r^2\dot{\theta}\,\hat{z}$$ (i.e., perp. to plane)

$\boxed{\text{Def. } \ell = \mu r^2\dot{\theta}}$ = constant of motion (lower-case is conventional notation)

$\Downarrow$

re-write $T = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2}$

$$\boxed{\Rightarrow E = T + U(r) = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2\mu}\frac{\ell^2}{r^2} + U(r) = \text{constant (i.e, conserved)}}$$

$\Downarrow$

$$\dot{r}^2 = \frac{2}{\mu}\left[E - U(r) - \frac{1}{2}\frac{\ell^2}{\mu r^2}\right] \Rightarrow \frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E-U(r)) - \frac{\ell^2}{\mu^2 r^2}}$$

$$\boxed{\Rightarrow t - t_0 = \int_{r_0}^{r}\frac{dr}{\sqrt{\frac{2}{\mu}(E-U(r)) - \frac{\ell^2}{\mu^2 r^2}}}}$$

can formally be inverted to find $r = r(t)$ $(*)$

Now consider angular coordinate: $d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr$

$$= \frac{\ell}{\mu r^2\dot{r}}dr$$

$$\Rightarrow d\theta = \frac{\ell/\mu r^2}{\sqrt{\frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}}}dr$$

$$\boxed{\Rightarrow \theta - \theta_0 = \int_{r_0}^{r}\frac{\ell/r^2}{\sqrt{2\mu\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}}dr}$$

$(**)$