PHY 852 - Notes on Chapter 8: Scattering at Lower Energies

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April 2022

We begin by writing the plane wave solutions in terms of the spherical Bessel Functions

$$e^{i\vec{k}\cdot\vec{r}} = \sum_l (2l+1) i^l j_l(kr) P_l(\cos\theta) \tag{1}$$

After manipulating and using the properties of the spherical Bessel functions, we end up with radial solutions for spherical symmetries

$$R_l(k,r) = \frac{1}{2}(e^{2i\delta_l} h(kr) + h_l^*(kr)) \tag{2}$$

Where δ_l is the phase shift due to the scattering and $h(kr)$ is spherical Henkel functions.

The wavefunction is then a linear combination of equation (1)

$$\psi_l(\vec{r}) = \sum_l (2l+1) i^l R_l(k,r) P_l(\cos\theta) \tag{3}$$

$$= e^{i\vec{k}\cdot\vec{r}} + \sum_l (2l+1)(R_l(k,r) - j_l(kr)) P_l(\cos\theta) \tag{4}$$

Computing $\psi_l(\vec{r})|_{r\to\infty}$

$$\psi_l(\vec{r})|_{r\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \sum_l (2l+1) e^{i\delta_l} \sin(\delta_l) \frac{e^{ikr}}{kr} P_l(\cos\theta) \tag{5}$$

We then define the scattering amplitude with units of length as

$$f(\Omega) \equiv \sum_l (2l+1) e^{i\delta_l} \frac{\sin(\delta_l)}{k} P_l(\cos\theta) \tag{6}$$

Our wavefunction then becomes

$$\psi_l(\vec{r})|_{r\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\Omega) \tag{7}$$

We can now relate the differential cross section to the flux of particles per solid angle, where v is the velocity and V is the volume.

$$\frac{v}{V}\frac{d\sigma}{d\Omega} = \frac{dN}{d\Omega dt} \tag{8}$$

$$\frac{v}{V}\frac{d\sigma}{d\Omega} = \frac{v}{V}\frac{|f(\omega)|^2}{r^2} \tag{9}$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)| \tag{10}$$

By solving this differential equation for the total cross section σ, we get

$$\boxed{\sigma = \frac{4\pi}{k^2}\sum_l (2l+1)\sin^2(\delta_l)} \tag{11}$$

It is important to know that for small energies $\sin(ka) \approx ka$. So the total cross section for small momenta is

$$\sigma = \frac{4\pi}{k^2}\sin^2(\delta) \approx 4\pi a^2 \tag{12}$$