August 2020 subject exam, Problem 5 Consider a ONE-DIMENSIONAL world, where a non-relativistic particle of mass M is in the ground state of a harmonic oscillator characterized by frequency ω_0. The harmonic oscillator is in a large box of length L which is populated by a bath of massless particles. The probability that any given state in the box is occupied is $f(k)$, where k is the wave number of the massless particle. The harmonic oscillator can be excited to the first excited state via the weak coupling,

$$V = g \int dx\, \Psi^\dagger(x)\, x\, \Psi(x)\Phi(x),$$

where Ψ is the field operator for the massive particle and $[\Psi(x,t), \Psi^\dagger(x',t)] = \delta(x-x')$, and Φ is the field operator for the massless particle,

$$\Phi(x,t) = \sum_k \frac{1}{\sqrt{2E_k L}} \left[a_k e^{-i\omega t + ikx} + a_k^\dagger e^{i\omega t - ikx} \right]$$

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}.$$

Using Fermi's golden rule, and using the dipole approximation, find the rate at which the massive particle is excited to the first excited state from the ground state. Your answer should be in terms of m, ω_0, g, and $f(k)$.

Solution.
One must first recognize that this problem can be solved using Fermi's golden rule, which is written as follows:

$$\Gamma = \frac{2\pi}{\hbar} \sum_k f(k) |\mathcal{M}|^2\, \delta(\epsilon_1 - \epsilon_0 - \epsilon_k) \tag{1}$$

Where $f(k)$ is the probability that any given momentum state in the box is occupied, ϵ_0 and ϵ_1 are the energies of the ground state and first excited state harmonic oscillator, respectively, and ϵ_k is the energy of the outgoing massless particle. $\mathcal{M}$ is the matrix element

$$\mathcal{M} = \langle \psi_1 | V | \psi_0, k \rangle, \tag{2}$$

where $|\psi_1\rangle$ is the final state and $|\psi_0, k\rangle$ is the initial state. We will begin by calculating this matrix element. To do this, we want to think about the contents of the coupling potential, V. One can see how the field operators act on the initial and final states.

$$\langle \psi_1 | \Psi^\dagger(x)\Psi(x)\Phi(x) | \psi_0, k \rangle = \psi_1^*(x)\psi_0(x)\, \langle 0 | \Phi(x) | k \rangle = \psi_1^*(x)\psi_0(x)\, e^{ikx} \frac{1}{\sqrt{2E_k L}}, \tag{3}$$

where $\psi_1(x)$ and $\psi_0(x)$ are the first excited harmonic oscillator and the ground state harmonic oscillator wavefunctions, respectively. In the above equation, the $\Psi^\dagger(x)$ and $\Psi(x)$ operators act on the single-particle states $|\psi_1\rangle$ and $|\psi_0\rangle$, respectively. They go by the following relation:

$$\Psi(x)\, |\psi_i\rangle = \psi_i(x)\, |0\rangle$$

The field operator for the massless particle, $\Phi(x)$, acts on the initial state, $|k\rangle$. Inspecting $\Phi(x,t)$, we notice that the $a_k^\dagger$ term does not contribute, so we leave off that part. This gives us the following:

$$\Phi(x)|k\rangle = e^{ikx}\frac{1}{\sqrt{2E_k L}}|0\rangle \tag{4}$$

Now, we can apply our above work to the integral for V. After plugging in equations 3 and 4 into equation 2, we get

$$\mathcal{M} = g\int dx\ e^{ikx}\psi_1^*(x)\,x\,\psi_0(x)\frac{1}{\sqrt{2E_k L}}.$$

Now is the time for our use of the dipole approximation. This approximation allows us to assume a small kx, or in other words, assuming that the emitted photon has a very long wavelength compared to the size of the emitter(atom). Maybe it should be named the "long wavelength approximation" instead! The dipole approximation, then, allows us to say

$$e^{ikx} \approx 1$$

Which, in turn, simplifies our matrix element integral!

$$\mathcal{M} = g\int dx\ \psi_1^*(x)\,x\,\psi_0(x)\frac{1}{\sqrt{2E_k L}}.$$

We can simplify our integral even further by realizing that we have the position operator, $\mathcal{X}$, sandwiched between single particle harmonic oscillator wave functions $\psi_1(x)$ and $\psi_0(x)$. This can be rewritten as $\langle 1|\mathcal{X}|0\rangle$. We can write the position operator in terms of the single particle creation and destruction operators that we are familiar with,

$$\mathcal{X} = \sqrt{\frac{\hbar}{2m\omega_0}}(a + a^\dagger)$$

Plugging this back into $\langle 1|\mathcal{X}|0\rangle$, we get

$$\langle 1|\mathcal{X}|0\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\ \langle 1|(a+a^\dagger)|0\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}$$

Now plugging this equation back into our matrix element, we get

$$\mathcal{M} = \frac{g}{\sqrt{2E_k L}}\sqrt{\frac{\hbar}{2m\omega_0}} \tag{5}$$

leading to the squared matrix element,

$$|\mathcal{M}|^2 = \frac{g^2\hbar}{4E_k L m\omega_0} \tag{6}$$

After finding the squared matrix element, we can insert it into Fermi's Goden Rule to estimate the decay rate.

$$\Gamma = \frac{g^2\pi}{2Lm\omega_0}\sum_k \frac{f(k)}{E_k}\delta(\hbar c|k| - \hbar\omega_0)$$

We can then transform the sum over momentum states to an integral, remembering to add a factor of $L/2\pi$ to account for the density of states, to get

$$\Gamma = \frac{g^2\pi}{2Lm\omega_0}\frac{L}{2\pi}\int_{-\infty}^{\infty}\frac{f(k)}{E_k}\delta(\hbar c|k| - \hbar\omega_0)dk = \frac{g^2}{4m\omega_0}\int_{-\infty}^{\infty}\frac{f(k)}{E_k}\delta(\hbar c|k| - \hbar\omega_0)dk$$

We then perform a change of variables to integrate over energy states instead. In this step, because $\hbar c|k| = \hbar\omega_0$ has two solutions, we add a factor of two. This accounts for a massless particle of momentum states of both -k and k.

$$\Gamma = \frac{g^2}{2m\omega_0}\int_0^{\infty}\frac{f(k)}{E_k}\frac{1}{dE_k/dk}\delta(E_k - \hbar\omega_0)dE_k$$

Simplifying,

$$\Gamma = \frac{g^2}{2m\omega_0\hbar c}\int_0^{\infty}\frac{f(k)}{E_k}\delta(E_k - \hbar\omega_0)dE_k$$

The integral evaluates to $(f(k)/E_k)|_{E_k=\hbar\omega_0}$, or equivalently, where $k = \omega_0/c$, so

$$\boxed{\Gamma = \frac{g^2}{2m\omega_0^2\hbar^2 c}f(\omega_0/c)} \tag{7}$$

You can check the units of g to be $[\text{mass}]^{1.5}$ $[\text{length}]^{2.5}$ $[\text{time}]^{-3}$, and verify that Γ has units of $[\text{time}]^{-1}$.