Chapter 1 Exam Review: States and Operators

April 18th, 2022
Physics 852

Basics of States and Operators

- Finite (e.g two component) systems, have a discrete number states, while other systems may have an infinite number. An orthonormal basis can be formed, any physical state of the system can be expressed as a linear combination of its members:

$$\langle\psi| = \sum_i a_i \langle i|$$

$$|\psi\rangle = \sum_i a_i^* |i\rangle$$

- The bra notation is shown first. Ket adjoint vectors are their complex transposes.
- States are normalized to have unit squares: $$\langle\psi|\psi\rangle = \sum_i a_i^* a_i = 1$$

- The squared overlap between two states gives the probability of observing a second state from an initial prepared state:

$$P = |\langle\psi|\phi\rangle|^2$$

- Operators act on functions/vectors to return transformed functions. We can express an operator in terms of its expectation for a pair of basis states:

$$A = \sum_{i,j} a_{i,j} |i\rangle \langle j|$$

Basics of Two Component Systems

- Consider the identity, and the Pauli matrices, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$:

$$\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

- Together, the set forms a complete, unitary basis for the space of 2x2 matrices. This basis in commonly used to describe spin-half systems (a two-component system in three space). Note, $\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$. Spin-up and spin-down states are expressed as:

$$|\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Rotations of such states in the spin-half system are expressed out by a Taylor expansion:

$$R(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{\sigma}/2} = e^{-i\theta\hat{n}\cdot\vec{\sigma}/2} = \cos(\theta/2) - i\sin(\theta/2)\vec{\sigma}\cdot\hat{n}$$

- Alternately, rotations of two-component systems in two space can be expressed: $R(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \cos\phi - i\sigma_y \sin\phi$
- One example of rotations in a 2D plane is the transformation of polarized fields about the direction of propagation (worth reviewing polarization naming conventions).

Example Problem 1: Two Component Neutrino Mixing

- A common application of two-state mixing is the atmospheric neutrino problem. Ground-based instruments have observed a deficit in the rate of muon neutrinos. As the rate of electron neutrinos was near expected, $\nu_\mu \to \nu_\tau$ must occur.
- We can introduce a Hamiltonian for these two observed states, and add in an additional mixing term:

$$H = \begin{bmatrix} m_\mu & 0 \\ 0 & m_\tau \end{bmatrix} + \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- This gives new Hamiltonian with different neutrino eigen-energies and eigen-states than those observed. **What are these new eigen-energies, and probabilities P(numu → nutau), P(numu → numu), as a function of time?**
- Begin by expressing ***H*** in terms of Pauli matrices:

$$\begin{aligned} H &= \begin{bmatrix} m_\mu & \alpha \\ \alpha & m_\tau \end{bmatrix} \\ &= \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} m_\mu + m_\tau & 0 \\ 0 & m_\mu + m_\tau \end{bmatrix} + \frac{1}{2} \begin{bmatrix} m_\mu - m_\tau & 0 \\ 0 & m_\tau - m_\mu \end{bmatrix} \\ &= \alpha\sigma_x + \frac{1}{2}(m_\mu + m_\tau)\mathbb{1} + \frac{1}{2}(m_\mu - m_\tau)\sigma_z. \end{aligned}$$

- To diagonalize, $\alpha\sigma_x + \frac{1}{2}(m_\mu + m_\tau)\mathbb{1} + \frac{1}{2}(m_\mu - m_\tau)\sigma_z$, note the triplet of Pauli matrices rotates as (x, y, x).
- Form an analogue vector of magnitude and direction from projections along the Pauli matrices:

$$\beta = \sqrt{\alpha^2 + (m_\mu - m_\tau)^2/4}$$

$$\hat{n} = (\alpha, 0, (m_\mu - m_\tau)/2)/\beta$$

$$H = \frac{1}{2}(m_\mu + m_\tau)\mathbb{1} + \beta(\vec{\sigma} \cdot \hat{n})$$

- Scalar eigen-energies are maintained under the transformation/rotation of Pauli matrices. Rotating along the (diagonalized) z direction,

$$\begin{aligned} H &= \frac{1}{2}(m_\mu + m_\tau)\mathbb{1} + \beta\sigma_z \\ &= \begin{bmatrix} (m_\mu + m_\tau)/2 + \beta & 0 \\ 0 & (m_\mu + m_\tau)/2 - \beta \end{bmatrix}. \end{aligned}$$

- Can now read eigen-energies for the neutrino mass eigenstates from the diagonal.

- The evolution operator for time-independent Hamiltonians is, $U = e^{-iHt/\hbar}$:
- Unitary transformation, so basis is transformed but maintained as orthonormal.

$$\begin{aligned} U(t) &= \exp\{-it(m_\mu + m_\tau)\mathbb{1}/(2\hbar) - it\beta(\vec{\sigma}\cdot\hat{n})/\hbar\} \\ &= \exp\{-it(m_\mu + m_\tau)/(2\hbar)\}(\cos(\beta t/\hbar)\mathbb{1} - i\vec{\sigma}\cdot\hat{n}\sin(\beta t/\hbar)) \\ &= \exp\{-it(m_\mu + m_\tau)/(2\hbar)\}(\cos(\beta t/\hbar)\mathbb{1} - (i/\beta)(\alpha\sigma_x + (m_\mu - m_\tau)\sigma_z/2)\sin(\beta t/\hbar)). \end{aligned}$$

- The probability of a specific oscillation is found by evolving the first state, and considering the squared projection onto the second:

$$\begin{aligned} \langle\nu_\tau|\, U(t)\, |\nu_\mu\rangle &= \begin{bmatrix} 0 & 1 \end{bmatrix} U(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \exp\{-it(m_\mu + m_\tau)/(2\hbar)\}(-(i/\beta)\alpha\sigma_x\sin(\beta t/\hbar)) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \exp\{-it(m_\mu + m_\tau)/(2\hbar)\}(-(i/\beta)\alpha\sin(\beta t/\hbar)) \end{aligned}$$

- The expectation of observing numu after time t is found similarly.

$$\begin{aligned} P(\nu_\mu \to \nu_\tau) &= \alpha^2\sin^2(\beta t/\hbar)/\beta^2, \\ P(\nu_\mu \to \nu_\mu) &= (m_\mu + m_\tau)^2\sin^2(\beta t/\hbar)/(4\beta^2) + \cos^2(\beta t/\hbar) \end{aligned}$$

- Nice overview of two flavor neutrino oscillation formalism: https://warwick.ac.uk/fac/sci/physics/staff/academic/boyd/warwick_week/neutrino_physics/lec_oscillations.pdf

Basics of Density Matrices

- States can be described by a density matrix:

$$\rho_\psi = |\psi\rangle\langle\psi|$$

- Density matrices are sufficient to generate all observables using:

$$\begin{aligned}\langle\psi|\mathcal{A}|\psi\rangle &= \sum_{i,j} \psi_i^* A_{ij} \psi_j \\ &= \mathrm{Tr}\, \rho_\psi \mathcal{A},\end{aligned}$$

$$\begin{aligned}|\langle\phi|\mathcal{A}|\psi\rangle|^2 &=< \psi|\mathcal{A}^\dagger|\phi\rangle\langle\phi|\mathcal{A}|\psi\rangle \\ &= \mathrm{Tr}\, \rho_\psi \mathcal{A}^\dagger \rho_\phi \mathcal{A}.\end{aligned}$$

- The trace of any product of matrices is invariant to unitary transformations, so the formulas above work regardless of basis (see example problem)

Example Problem 2: Density Matrices

Fall 1998 Final

1. (15 pt.s) Consider a spin 1/2 system. The projection operator P_z projects the component of the wave function that has positive spin along the z axis.

$$\langle \eta | P_z | \eta \rangle = |\langle z, \uparrow | \eta \rangle|^2$$

(a) Express P_z as a matrix in the basis where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ denotes a state with positive spin along the z axis.

(b) Write down the density matrix for a state that is an incoherent mixture of 50% positive spin along the y axis and 50% negative spin along the y axis.

(c) If the Hamiltonian is defined as:

$$\mathcal{H} = \alpha + \beta \sigma_x$$

Calculate the expectation of $\mathcal{H}$ for the state described in b.

(a) Express P_z as a matrix in the basis where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ denotes a state with positive spin along the z axis.

$$|z,\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |z,\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$P_z = |z,\uparrow\rangle\langle z,\uparrow| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\boxed{P_z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$$

Check: $\langle \eta | P_z | \eta \rangle = |\langle z,\uparrow | \eta \rangle|^2$

$$|\eta\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\langle \eta | P_z | \eta \rangle = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a^2$$

$$\langle z,\uparrow | \eta \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a$$

$$|\langle z,\uparrow | \eta \rangle|^2 = a^2$$

(b) Write down the density matrix for a state that is an incoherent mixture of 50% positive spin along the y axis and 50% negative spin along the y axis.

$$\rho = \frac{1}{2}|y,\uparrow\rangle\langle y,\uparrow| + \frac{1}{2}|y,\downarrow\rangle\langle y,\downarrow|$$

2 ways:

$$|y,\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |y,\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\rho = \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$$

$$\boxed{\rho = \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}}$$

$$|y,\uparrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \qquad |y,\downarrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

$$\rho = \frac{1}{2}\cdot\frac{1}{2}\begin{pmatrix}1\\i\end{pmatrix}\begin{pmatrix}1 & -i\end{pmatrix} + \frac{1}{2}\cdot\frac{1}{2}\begin{pmatrix}1\\-i\end{pmatrix}\begin{pmatrix}1 & i\end{pmatrix}$$

$$\rho = \frac{1}{4}\begin{pmatrix}1 & -i\\i & 1\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1 & i\\-i & 1\end{pmatrix}$$

$$\rho = \frac{1}{4}\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$$

$$\boxed{\rho = \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}}$$

(c) If the Hamiltonian is defined as:

$$\mathcal{H} = \alpha + \beta\sigma_x$$

Calculate the expectation of $\mathcal{H}$ for the state described in b.

Short way: $\langle H \rangle = \mathrm{Tr}(\rho H)$

$$\mathrm{Tr}(\rho H) = \alpha \mathrm{Tr}(\rho) + \beta \mathrm{Tr}(\rho \sigma_x)$$

$$\mathrm{Tr}(\rho H) = \alpha + \frac{\beta}{2} \mathrm{Tr}(\sigma_x)$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\boxed{\langle H \rangle = \alpha}$$

(c) If the Hamiltonian is defined as:

$$\mathcal{H} = \alpha + \beta\sigma_x$$

Calculate the expectation of $\mathcal{H}$ for the state described in b.

Long way:

$$H = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

$$\langle H \rangle = \frac{1}{2}\langle y, \uparrow |H| y, \uparrow\rangle + \frac{1}{2}\langle y, \downarrow |H| y, \downarrow\rangle|$$

$$\langle H \rangle = \frac{1}{4}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & i \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\langle H \rangle = \frac{1}{4}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} \alpha + i\beta \\ \beta + i\alpha \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & i \end{pmatrix}\begin{pmatrix} \alpha - i\beta \\ \beta - i\alpha \end{pmatrix}$$

$$\langle H \rangle = \frac{1}{2}\alpha + \frac{1}{2}\alpha$$

$$\boxed{\langle H \rangle = \alpha}$$