PHY852 Chapter 14 Review: States Without Conserved Particle Number

Amelia Doetsch, Elizabeth Kowalczyk

Department of Physics and Astronomy - Michigan State University

Spring 2022

Introduction: States Without Conserved Particle Number
The main focus of this section is coherent states. They are typically denoted as $|\eta\rangle$ to distinguish them from wave functions with fixed particle number (which *typically* look like $|\psi\rangle$, $|\phi\rangle$, $|k_p\rangle$, $|n\rangle$, etc.). $|\alpha\rangle$, $|\beta\rangle$, and $|\gamma\rangle$ have also been used in past subject exams. Some useful properties of coherent states are:

- A coherent state is a linear combination of states with different particle numbers and can be written in the general form:

$$|\eta\rangle = e^{-\eta^*\eta/2}e^{\eta a^\dagger}|0\rangle = e^{\eta a^\dagger - \eta^* a}|0\rangle = e^{-\eta^*\eta/2}\sum_n \frac{(\eta a^\dagger)^n}{n!}|0\rangle$$

- η is a complex number.
- Coherent states provide a complete basis:

$$\frac{1}{\pi}\int d\eta_R d\eta_I \langle m|\eta\rangle\langle\eta|n\rangle = \delta_{mn}$$

- A coherent state is an eigenstate of the destruction operator. That is:

$$a|\eta\rangle = \eta|\eta\rangle$$

- Coherent states are solutions to the Hamiltonian:

$$H(t) = H_0 + V(t) = \epsilon a^\dagger a + j(t)(a^\dagger + a)$$

  where $j(t)$ is like an external classical current.
- In the interaction representation:

$$|\eta(t)\rangle_I = e^{\eta(t)a^\dagger - \eta^*(t)a}|0\rangle$$

  with the normalized state being:

$$\eta(t) = \frac{-i}{\hbar}\int_{-\infty}^{t} dt' e^{i\epsilon t'/\hbar} j(t')$$

These are some of the most important things to know about chapter 14 on the exam. There is plenty more to think about in the lecture notes, but an understanding of the first four bullet points may be sufficient for the subject exam. **None of these equations are on past equation sheets.**

Possible questions include:

- Find the inner product between coherent states. (August 2021 Subject Exam Problem 2, Spring 2020 Subject Exam Problem 1a, Problem 14.3)

- Find the transition matrix element of a ladder operator with coherent states. (Spring 2001 Subject Exam Problem 7, Problem 14.4, Spring 2020 Subject Exam Problem 1c)
- Find the expectation value of the energy with coherent state solutions to the time dependent Hamiltonian (Spring 2020 Subject Exam Problem 1b)

Example Problem: August 2021 Subject Exam, Problem 2 Consider the states:

$$|\eta\rangle = e^{i(\eta a^\dagger + \eta^\star a)} |0\rangle$$

$$|\gamma\rangle = e^{i(\gamma a^\dagger + \gamma^\star a)} |0\rangle$$

Here η and γ are complex numbers, and $a^\dagger$ and a are creation and annihilation operators respectively.

Problem 1
Calculate $\langle\eta|\eta\rangle$.

The key to the problem is to realize that $(\eta a^\dagger + \eta^\star a)$ is hermitian, while i is not. So, the exponential's complex conjugate is its inverse.

$$\langle\eta|\eta\rangle = \langle 0| e^{-i(\eta a^\dagger + \eta^\star a)} e^{i(\eta a^\dagger + \eta^\star a)} |0\rangle = 1$$

Problem 2
Calculate $\langle\gamma|\eta\rangle$.

We will start this problem by simplifying $|\eta\rangle$ and $|\gamma\rangle$ by a rearranged Baker Campbell Hausdorff Theorem.

$$e^A e^B e^{-[A,B]/2} = e^{A+B}$$

So,

$$e^{i(\eta a^\dagger + \eta^\star a)} = e^{i\eta a^\dagger} e^{i\eta^\star a} e^{-[i\eta a^\dagger, i\eta^\star a]/2}$$

And we know $[a^\dagger, a] = 1$, and commutators are antisymmetric.

$$= e^{i\eta a^\dagger} e^{i\eta^\star a} e^{-i^2|\eta|^2(-1)/2} = e^{i\eta a^\dagger} e^{i\eta^\star a} e^{-|\eta|^2/2}$$

We can then see that when attached to the $|0\rangle$, the $e^{i\eta^\star a}$ goes to 1, as the only term of its taylor expansion that survives in the leading order. This leads us to:

$$|\eta\rangle = e^{-|\eta|^2/2} e^{i\eta a^\dagger} |0\rangle$$

$$|\gamma\rangle = e^{-|\gamma|^2/2} e^{i\gamma a^\dagger} |0\rangle$$

Then, taking the inner product,

$$\langle\gamma|\eta\rangle = \langle 0| e^{-|\gamma|^2/2} e^{-i\gamma^\star a} e^{-|\eta|^2/2} e^{i\eta a^\dagger} |0\rangle$$
$$= e^{-(|\eta|^2+|\gamma|^2)/2} \langle 0| e^{-i\gamma^\star a} e^{i\eta a^\dagger} |0\rangle$$

We can then use the Taylor series to express the exponentials

$$= e^{-(|\eta|^2+|\gamma|^2)/2} \langle 0| \sum_m \frac{(\gamma^\star a)^m}{m!} \sum_n \frac{(\eta a^\dagger)^n}{n!} |0\rangle$$

From this we can see that when $m \neq n$, the term is 0.

$$= e^{-(|\eta|^2+|\gamma|^2)/2} \langle 0| \sum_n \frac{(\gamma^\star \eta)^n}{(n!)^2} (aa^\dagger)^n |0\rangle$$

And we know that $aa^\dagger$ is the number operator, applied n number of times ($= n!$).

$$= e^{-(|\eta|^2+|\gamma|^2)/2} \langle 0| \sum_n \frac{(\gamma^\star \eta)^n}{n!} |0\rangle = e^{-(|\eta|^2+|\gamma|^2)/2} e^{\gamma^\star \eta}$$

Example Problem: Spring 2001 Subject Exam, Problem 7
Consider the quantum state:

$$|\eta\rangle = e^{-\eta^* \eta/2} e^{\eta a^\dagger} |0\rangle$$

and note that:

$$\langle\eta|\eta\rangle = 1$$

Problem 1
Calculate $\langle 0| a |\eta\rangle$.

This problem becomes simple when you realize that $|\eta\rangle$ is an eigenstate of the destruction operator:

$$\langle 0| a |\eta\rangle$$
$$= \eta \langle 0|\eta\rangle$$
$$= \eta e^{-\eta^*\eta/2} \langle 0| e^{\eta a^\dagger} |0\rangle$$
$$= \eta e^{-\eta^*\eta/2} \langle 0| (1 + \eta a^\dagger + ...) |0\rangle$$
$$= \eta e^{-\eta^*\eta/2}$$

Problem 2
Calculate $\langle \eta| (a^\dagger)^3 a^2 |\eta\rangle$.

Again, the key to this problem is that $|\eta\rangle$ is an eigenstate of the destruction operator. Note that:

$$\langle \eta| (a^\dagger)^3$$
$$= (a^3 |\eta\rangle)^\dagger$$

Noting that a single $a|\eta\rangle$ doesn't change what's inside the ket, you can just pull out one η for each a:

$$= (\eta^3 |\eta\rangle)^\dagger$$
$$= (\eta^*)^3 \langle \eta|$$

It is then simple to see that our matrix element is:

$$\langle \eta| (a^\dagger)^3 a^2 |\eta\rangle$$
$$= (\eta^*)^3 \eta^2 \langle \eta|\eta\rangle$$
$$= (\eta^*)^3 \eta^2$$