Chapter 4

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Chapter 4-> lot of smaller things (4 homeworks)

Chapter 4 covers many topics such as rotations, symmetries, raising and lowering operators, and the hydrogen atom. We will review these topics briefly, then cover an example problem that has been on past exams

“You can just inspect the answer” -Scottpocket

Rotation

- 3D rotation is just decomposed into 2 2d rotations

$$e^{i\vec{L}\cdot\vec{\beta}/\hbar}e^{i\vec{L}\cdot\vec{\alpha}/\hbar} = e^{i\vec{L}\cdot\vec{\gamma}/\hbar}.$$

$$\begin{aligned}
e^{i\vec{\beta}\cdot\vec{\sigma}/2} &= \cos(\beta/2) + i(\hat{\beta}\cdot\vec{\sigma})\sin(\beta/2), \\
e^{i\vec{\alpha}\cdot\vec{\sigma}/2} &= \cos(\alpha/2) + i(\hat{\alpha}\cdot\vec{\sigma})\sin(\alpha/2), \\
e^{i\vec{\beta}\cdot\vec{\sigma}/2}e^{i\vec{\alpha}\cdot\vec{\sigma}/2} &= \cos(\beta/2)\cos(\alpha/2) + i(\hat{\beta}\cdot\vec{\sigma})\sin(\beta/2)\cos(\alpha/2) \\
&+ i(\hat{\alpha}\cdot\vec{\sigma})\sin(\alpha/2)\cos(\beta/2) \\
&- (\hat{\beta}\cdot\vec{\sigma})(\hat{\alpha}\cdot\vec{\sigma})\sin(\beta/2)\sin(\alpha/2), \\
&= \cos(\gamma/2) + i(\hat{\gamma}\cdot\vec{\sigma})\sin(\gamma/2).
\end{aligned}$$

$$\cos(\gamma/2) = \cos(\beta/2)\cos(\alpha/2) - \hat{\beta}\cdot\hat{\alpha}\sin(\alpha/2)\sin(\beta/2).$$

$$\begin{aligned}
\sin(\gamma/2)\hat{\gamma} &= \hat{\beta}\sin(\beta/2)\cos(\alpha/2) + \hat{\alpha}\sin(\alpha/2)\cos(\beta/2) \\
&- (\hat{\beta}\times\hat{\alpha})\sin(\beta/2)\sin(\alpha/2).
\end{aligned}$$

Raising and Lowering operators

In terms of normalized states, the raising and lowering operators acting on a state $|m\rangle$ become

$$\boldsymbol{L}_{\pm}|m\rangle = \hbar[\ell(\ell+1) - m^2 \mp m]^{1/2}|m \pm 1\rangle. \tag{4.23}$$

Hydrogen Atom

$$-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}u(r)+\left(\frac{\hbar^2\ell(\ell+1)}{2\mu r^2}+V(r)\right)u(r)=Eu(r).$$

$$u_\ell(r)\equiv r\phi_\ell(r).$$

$$V(r)=-\frac{e^2}{r}.\qquad E_n=-\frac{e^2}{2a_0}\frac{1}{n^2}.$$

$$\begin{aligned}
L_1^1(x) &= -1\\
L_2^1(x) &= 2x-4\\
L_2^2(x) &= 2\\
L_3^1(x) &= -3x^2+18x-18\\
L_3^2(x) &= -6x+18\\
L_3^3(x) &= -6\\
L_4^1(x) &= 4x^3-48x^2+144x-96\\
L_4^2(x) &= 12x^2-96x+144\\
L_4^3(x) &= 24x-96\\
L_4^4(x) &= 24
\end{aligned}$$

$$R_{n,\ell}(r)=\frac{u_{n,\ell}}{r}=\left\{\left(\frac{2}{na_0}\right)^3\frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}\right\}^{1/2}e^{-r/(na_0)}\left(\frac{2r}{na_0}\right)^\ell L_{n+\ell}^{2\ell+1}\left(\frac{2r}{na_0}\right).$$

Neutron and Proton Spin-Spin and Magnetic Interaction

A neutron and proton are each in an s wave of a nuclear potential. The two particles feel a spin-spin interaction,

$$V_{ss} = -\alpha \vec{S}_p \cdot \vec{S}_n.$$

The nucleons also are in region with external magnetic field of strength B,

$$V_b = -\mu_p \vec{B} \cdot \vec{S}_p - \mu_n \vec{B} \cdot \vec{S}_n.$$

In the J, M basis,

$V_{SS} = -(\alpha\hbar^2/2)(S(S+1) - 3/2))$

$$V_{ss} = \begin{pmatrix} -\alpha\hbar^2/4 & 0 & 0 & 0 \\ 0 & -\alpha\hbar^2/4 & 0 & 0 \\ 0 & 0 & -\alpha\hbar^2/4 & 0 \\ 0 & 0 & 0 & 3\alpha\hbar^2/4 \end{pmatrix}$$

So the eigenvalues for the spin spin interaction are easy to find in this basis

To find the energies of the magnetic interaction, we must rewrite the states in this basis

$$\begin{aligned}
|j=1,m=1\rangle &= |m_p=1/2,m_n=1/2\rangle, \\
|j=1,m=-1\rangle &= |m_p=-1/2,m_n=-1/2\rangle, \\
|j=1,m=0\rangle &= \frac{1}{\sqrt{2}}\left(|m_p=1/2,m_n=-1/2+|m_p=-1/2,m_n=1/2\rangle\right), \\
|j=0,m=0\rangle &= \frac{1}{\sqrt{2}}\left(|m_p=1/2,m_n=-1/2\rangle-|m_p=-1/2,m_n=1/2\rangle\right) .
\end{aligned}$$

We can see the mixing from these last two states in the matrix

$$V_b=-\frac{\hbar B}{2}\begin{pmatrix} (\mu_p+\mu_n) & 0 & 0 & 0 \\ 0 & -(\mu_p+\mu_n) & 0 & 0 \\ 0 & 0 & 0 & (\mu_p-\mu_n) \\ 0 & 0 & (\mu_p-\mu_n) & 0 \end{pmatrix}.$$

Finding the eigenvalues, the first two terms are diagonalized, so

$$V_{ss} = \begin{pmatrix} -\alpha\hbar^2/4 & 0 & 0 & 0 \\ 0 & -\alpha\hbar^2/4 & 0 & 0 \\ 0 & 0 & -\alpha\hbar^2/4 & 0 \\ 0 & 0 & 0 & 3\alpha\hbar^2/4 \end{pmatrix} \qquad V_b = -\frac{\hbar B}{2}\begin{pmatrix} (\mu_p+\mu_n) & 0 & 0 & 0 \\ 0 & -(\mu_p+\mu_n) & 0 & 0 \\ 0 & 0 & 0 & (\mu_p-\mu_n) \\ 0 & 0 & (\mu_p-\mu_n) & 0 \end{pmatrix}.$$

Finding the eigenvalues, the first two terms are diagonalized, so

$$\epsilon_1 = -\alpha\hbar^2/4 + (\mu_p+\mu_n)\hbar B/2$$
$$\epsilon_2 = -\alpha\hbar^2/4 - (\mu_p+\mu_n)\hbar B/2,$$

And we find the second two by puting the 2x2 matrix in terms of spin matrices, yielding the remaining two eigenvalues

$$\epsilon_\pm = \alpha\hbar^2/4 \pm \sqrt{\alpha^2\hbar^4/4 + (\mu_p-\mu_n)^2\hbar^2B^2/4}$$

Just some extra stuff

$$e^{A} e^{B} = e^{A+B} e^{C/2} \qquad C = [A, B]$$

$$\mathcal{R}_i \mathcal{R}_j = \mathcal{R}_k$$

$$L_z |\ell, m\rangle = m\hbar |\ell, m\rangle\,, \quad L^2 |\ell, m\rangle = \ell(\ell+1)\hbar^2 |\ell, m\rangle.$$

$$\Psi(\vec{r}) = \phi_\ell(r) Y_{\ell,m}(\theta, \phi)$$

- Lowest energy of spherically symmetric-> pages 80-81