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1. **Fermi Gas**

Imagine a 2D world with three nonrelativistic spinless particle species a, b, c and mass m. These particles can interact via

$$a + b \leftrightarrow c + \gamma.$$

These species' number densities follow the constraints $n = n_a + n_c$ and $n_a = n_b$ where n is a constant.
Find the fraction $\frac{n_c}{n}$ at equilibrium.

Solution:

$$n_x = \frac{1}{4\pi^2}\int_{k<k_x} d^2k = \frac{k_x^2}{4\pi}$$

$$\varepsilon_x = \frac{\hbar^2 k_x^2}{2m}$$

where $x = a, b, c$.

$$n_a = n_b \implies k_a = k_b$$

$$n = n_a + n_c \implies 4\pi n = k_a^2 + k_c^2$$

At equilibrium we have,

$$\varepsilon_a + \varepsilon_b = \varepsilon_c \implies k_a^2 + k_b^2 = k_c^2.$$

This follows from the fact that with an energy imbalance, e.g. $\varepsilon_a + \varepsilon_b > \varepsilon_c$, it will be energetically favorable for a, b particles to produce some number of c particles.

$$\implies 2k_a^2 = k_c^2 \implies 2(4\pi n - k_c^2) = k_c^2$$

$$\implies k_c^2 = \frac{8\pi}{3}n \implies \frac{n_c}{n} = \frac{2}{3}$$

This is the simplest possible form of a problem like this. Complications on an exam may be:

(a) Different dimensionality (1, 3)
(b) Increased degrees of freedom (e.g. color for quarks)
(c) Different rules for charge conservation (e.g. every species has charge, different charge ratios)
(d) Make one or more particle massless
(e) Different masses for species that have it

2. **Magnetic Susceptibility**

Electrons are confined to a two-dimensional surface to move in the x-y plane ($z = p_z = 0$). The areal density of electrons, number of electrons per area is σ. Electrons of the same spin have the same energy until a magnetic field is added along the z-axis. This gives the interaction

$$H_B = g_s \mu_B B \frac{s_z}{\hbar}$$

with $\mu_B = \frac{e\hbar}{2mc}$ and $g_s = 2$.
a. In terms of m and σ, what is the Fermi energy and Fermi wave number when $B = 0$?
b. What is the aerial magnetic moment density, M_z, for small fields?

$$M_z = \frac{1}{2} g_s \mu_B (\sigma_\downarrow - \sigma_\uparrow) = \chi B$$

Solution:
a. To find σ,

$$\sigma = \frac{g_s}{(2\pi)^d} \int_{k<k_f} d^d k$$

Then,

$$\sigma = \frac{1}{2\pi} k_F^2$$
$$k_F = \sqrt{2\pi\sigma}$$

We know the energies are $E = \frac{k^2\hbar^2}{2m}$, so

$$E_F = \frac{\pi\sigma\hbar^2}{m}$$

b. From before, we write the aerial densities as

$$\sigma_\uparrow = \frac{1}{2\pi} k_\uparrow^2, \sigma_\downarrow = \frac{1}{2\pi} k_\downarrow^2$$

So,

$$\sigma_\downarrow - \sigma_\uparrow = \frac{1}{2\pi}(k_\downarrow^2 - k_\uparrow^2) = \frac{1}{2\pi} k_f \Delta k_F$$

Where we have used that $(k_\downarrow - k_\uparrow) = \Delta k_F$ and $(k_\downarrow + k_\uparrow) = k_F$.
Now we note that

$$\Delta E_F = \frac{\hbar^2}{2m} k_F \Delta k_F$$
$$\Rightarrow \Delta k_F = \frac{2m\Delta E}{\hbar^2 k_F}$$

However, we are given that $H_B = g_s \mu_B B \frac{s_z}{\hbar}$, which gives

$$\Delta E_F = g_s \mu_B B$$

So

$$\Delta k_F = \frac{4m\mu_B}{\hbar^2 k_F^2} B$$

Plugging everything into the expression for M_z,

$$M_z = \frac{2m\mu_B^2}{\pi\hbar^2} B$$

Giving

$$\chi = \frac{2m\mu_B^2}{\pi\hbar^2} = \frac{e^2}{2\pi mc^2}$$