Chapter 8 – Homework Solutions

1. Show that if the function $u_\ell(kr)$ is defined in terms of $R_\ell(r)$

$$u_\ell(kr) \equiv r R_\ell(r),$$

where R_ℓ is a solution to the radial Schrödinger equation

$$\left\{ -\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2} + V(r) \right\} R_\ell(r) = \frac{\hbar^2 k^2}{2m} R_\ell(r),$$

that u_ℓ satisfies the differential equation,

$$\left(\frac{d^2}{dx^2} + 1 \right) u_\ell(x) = \frac{\ell(\ell+1)}{x^2} u_\ell(x) + \beta(x) u_\ell(x),$$

where β is proportional to the potential,

$$\beta(x) = \frac{2m}{\hbar^2 k^2} V(x/k).$$

Solution:

$$\begin{aligned}
r\left\{ -\frac{\hbar^2}{2m}\frac{1}{r}\partial_r^2 r + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2} + V(r) \right\} \frac{u_\ell}{r} &= \frac{\hbar^2 k^2}{2m} u_\ell, \\
\left\{ -\frac{\hbar^2}{2m}\partial_r^2 + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} + V(r) \right\} u_\ell &= \frac{\hbar^2 k^2}{2m} u_\ell, \\
\left\{ \partial_r^2 + \frac{\ell(\ell+1)}{r^2} + \frac{2mV(r)}{\hbar^2} \right\} u_\ell &= k^2 u_\ell(r), \\
x &= kr, \\
\left\{ -\partial_x^2 + \frac{\ell(\ell+1)}{x^2} + \beta(x) \right\} u_\ell = u_\ell. \quad \checkmark
\end{aligned}$$

2. Recurrence relations for Bessel functions provide you the ability to find forms for solutions at higher ℓ given you know the form for $\ell = 0$ and $\ell = 2$

(a) Show that in the case of zero potential that the solutions u_ℓ satisfy the recurrence relation.

$$u_{\ell+1}(x) = \frac{(\ell+1)}{x} u_\ell(x) - \frac{d}{dx} u_\ell(x).$$

Use the expressions from the previous problem,

$$\left(\frac{d^2}{dx^2} + 1\right) u_\ell(x) = \frac{\ell(\ell+1)}{x^2} u_\ell(x) + \beta(x) u_\ell(x). \quad (1)$$

(b) Show that this recurrence relation can be equivalently expressed as

$$f_{\ell+1}(x) = \frac{\ell}{x} f_\ell(x) - \frac{d}{dx} f_\ell(x),$$

where f_ℓ is a solution to the radial Schrödinger equation, $f_\ell(kr) \equiv u_\ell(kr)/(kr)$, which means that f_ℓ might be any linear combination of j_ℓ and n_ℓ.

(c) One can also show that a second recurrence relation is satisfied,

$$f_{\ell-1}(x) = \frac{(\ell+1)}{x} f_\ell(x) + \frac{d}{dx} f_\ell(x).$$

Given this recurrence relation, plus the one from the previous problem, show that

$$f_{\ell-1}(x) + f_{\ell+1}(x) = \frac{(2\ell+1)}{x} f_\ell(x)$$

(d) Using expressions for j_0, j_1, n_0 and n_1, use recurrence relations to find expressions for j_2 and n_2.

(e) Using the recurrence relations, show that $j_\ell(z)$ and $n_\ell(z)$ behave as z^ℓ and $z^{-(\ell+1)}$ respectively for $z \to 0$. Begin with the facts that $j_0(z)$ and $n_0(z)$ behave as z^0 and z^{-1} respectively, and that they are even and odd functions in z.

Solution:
a) Begin by inserting the expression for $u_{\ell+1}$ to see if it satisfies the differential equation for $\ell+1$.

$$\left[-\partial_x^2 + \frac{(\ell+1)(\ell+2)}{x^2} - 1\right]\left[\frac{(\ell+1)}{x} u_\ell - \partial_x u_\ell\right] =?0,$$

$$\left[-\partial_x^2 + \frac{(\ell)(\ell+1)}{x^2} - 1 + \frac{(2\ell+2)}{x^2}\right]\left[\frac{(\ell+1)}{x} u_\ell - \partial_x u_\ell\right] =?0,$$

$$-\frac{2(\ell+1)}{x^3} u_\ell + \frac{2(\ell+1)}{x^2}\partial_x u_\ell + \frac{(2\ell+2)(\ell+1)}{x^3} u_\ell$$

$$+\partial_x\left[\left(\frac{\ell(\ell+1)}{x^2} - 1\right) u_\ell\right] - \left[\frac{\ell(\ell+1)}{x^2} - 1\right]\partial_x u_\ell - \frac{2(\ell+1)}{x^2}\partial_x u_\ell =?0,$$

Eq. (1) was used to eliminated the term with $\partial_x^2 u_\ell$.

$$\left[\frac{2(\ell+1)}{x^2} - \frac{2(\ell+1)}{x^2}\right]\partial_x u_\ell + \left[-\frac{2(\ell+1)}{x^3} + \frac{2(\ell+1)^2}{x^3} - \frac{2\ell(\ell+1)}{x^3}\right]u_\ell =?0$$

One can see that both terms on the l.h.s. are zero.

b)

$$\begin{aligned}
u_{\ell+1} &= \frac{\ell+1}{x}u_\ell - \partial_x u_\ell,\\
x f_{\ell+1} &= \frac{(\ell+1)}{x}x f_\ell - \partial_x(x f_\ell),\\
f_\ell + 1 &= \frac{\ell+1}{x}f_\ell - \frac{1}{x}f_\ell - \partial_x f_\ell,\\
&= \frac{\ell}{x}f_\ell - \partial_x f_\ell \quad \checkmark
\end{aligned}$$

c) Add the expressions for $f_{\ell-1}$ and $f_{\ell+1}$,

$$\begin{aligned}
f_{\ell-1} + f_{\ell+1} &= \left(\frac{\ell}{x} + \frac{(\ell+1)}{x}\right)f_\ell\\
&= \frac{(2\ell+1)}{x}f_\ell \quad \checkmark
\end{aligned}$$

d) Use the relation:

$$\begin{aligned}
f_{\ell-1} + f_{\ell+1} &= \frac{(2\ell+1)}{x}f_\ell,\\
f_{\ell+1} &= \frac{(2\ell+1)}{x}f_\ell - f_{\ell-1}.
\end{aligned}$$

$$\begin{aligned}
j_2 &= -j_0 + \frac{3}{x}j_1\\
&= -\frac{\sin x}{x} - \frac{3}{x^2}\cos x + \frac{3}{x^2}\sin x,\\
n_2 &= -n_0 + \frac{3}{x}n_1,\\
&= \frac{\cos x}{x} - \frac{3}{x^3}\cos x - \frac{3}{x^2}\sin x.
\end{aligned}$$

e) Start with low z behavior for $j_0(z)$ and $n_0(z)$.

$$j_0 \sim z^0, \quad n_0 \sim z^{-1}, \; f_{\ell+1} \qquad = \frac{\ell}{z}f_\ell - \frac{d}{dz}f_\ell.$$

Assume $j_\ell \sim z^\ell$ for some ℓ.

$$\begin{aligned}
j_{\ell+1} &= \frac{\ell}{z}j_\ell - \partial_z j_\ell,\\
j_\ell &= Az^\ell + z^{\ell+2} + \cdots,\\
j_{\ell+1} &= A\ell z^{\ell-1} - B(\ell+2)z^{2\ell+1} - A_\ell \ell z^{\ell-1} + B\ell z^{\ell+1} + \cdots\\
&= 2Bz^{\ell+1}.
\end{aligned}$$

Thus, this must work for all ℓ.
Now, doe the same for n_ℓ. Assume that for some ℓ

$$n_\ell = Az^{-(\ell+1)} + B^{-(\ell-1)} + \cdots ,$$

The recurrence relation leads to

$$\begin{aligned} n_{\ell+1} &= \frac{\ell}{z} n)\ell - \partial_z n_\ell, \\ &= \ell A z^{-(\ell+2)} + (\ell+1) A z^{-(\ell+2)} + \cdots \\ &= (2\ell+1) A z^{-((\ell+1)+2} = (2\ell+1) A z^{-\ell+1}. \end{aligned}$$

Thus, this works for all ℓ

3. Consider a particle of mass m that interacts with a spherically symmetric attractive potential.

$$V(r) = \begin{cases} -V_0, & r < b \\ 0, & r > b \end{cases}$$

(a) What is the minimum depth $V_{\min}$ that allows a bound state?

(b) Find an expression for the phase shift in terms of a particle whose momentum is p.

(c) Assuming the depth is $V_0 = 0.99 \cdot V_{\min}$, plot the s-wave phase shift for momenta in the range $0 < p < 5\hbar/b$. Use units of $\hbar/b$ for the momenta.

(d) Repeat the above problem for $V_0 = 1.01 \cdot V_{\min}$.

(e) What are the scattering lengths for the two potentials?

Solution:
a)

$$\psi_I = A\sin(k_I r), \quad \psi_{II} = e^{-qr},$$
$$-V_0 + \frac{\hbar^2 k_I^2}{2m} = -\frac{\hbar^2 q^2}{2m}.$$

For barely bound state, $q \to 0$ and

$$k_I = \sqrt{2mV_0/\hbar^2}.$$

If ψ_I in this limit is to match to exponential wave function with $q = 0$, it must have zero slope. Thus

$$\begin{aligned} k_I b &= \pi/2, \\ \sqrt{\frac{2mV_0}{\hbar^2}} b &= \pi/2, \\ V_0 &= \frac{\pi^2\hbar^2}{8mb^2}. \end{aligned}$$

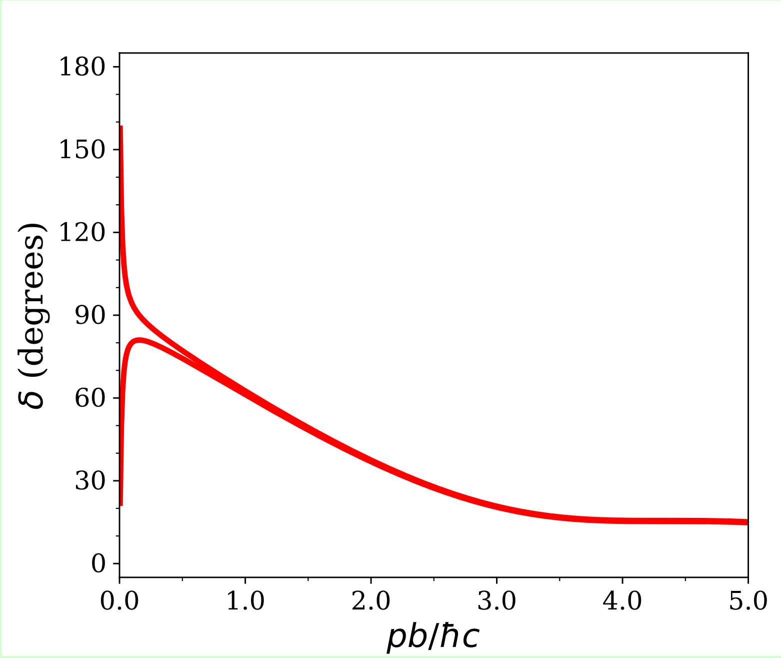
b) For scattering, the wave function in region II is

$$\psi_{II} = \sin(kr + \delta)$$

$$\begin{aligned} A\sin(k_I b) &= A\sin(kb + \delta), \\ k_I A\cos(k_I b) &= k\cos(kb + \delta), \\ \frac{k}{k_I}\tan(k_I B) &= \tan(kb + \delta), \end{aligned}$$

$$\begin{aligned} \delta &= -kb + \arctan\left(\frac{k}{k_I}\tan(k_I b)\right), \\ &= -\frac{pb}{\hbar} + \arctan\left(\frac{p}{q}\tan(qb/\hbar)\right), \\ q &= \sqrt{2mV_0 + p^2}. \end{aligned}$$

c) and d)



e) Take the expression for δ for small p,

$$\begin{aligned}\delta &= -kb + \arctan\left(\frac{k}{k_I}\tan(k_I b)\right) \\ &\approx -kb + k\frac{\tan(k_I b)}{k_I}, \\ &= -\frac{pb}{\hbar} + p\frac{\tan(\sqrt{2mV_0/\hbar^2}b)}{\sqrt{2mV_0}}.\end{aligned}$$

The scattering length is then

$$\ell = -b + \frac{\tan(\sqrt{2mV_0/\hbar^2}b)}{\sqrt{2mV_0/\hbar^2}}$$

The scattering lengths change from $+\infty$ to $-\infty$ when the argument of the tangent crosses $\pi/2$. This is the same condition as having the bound state disapper.

4. Consider a proton scattering off of a an attractive one-dimensional potential,

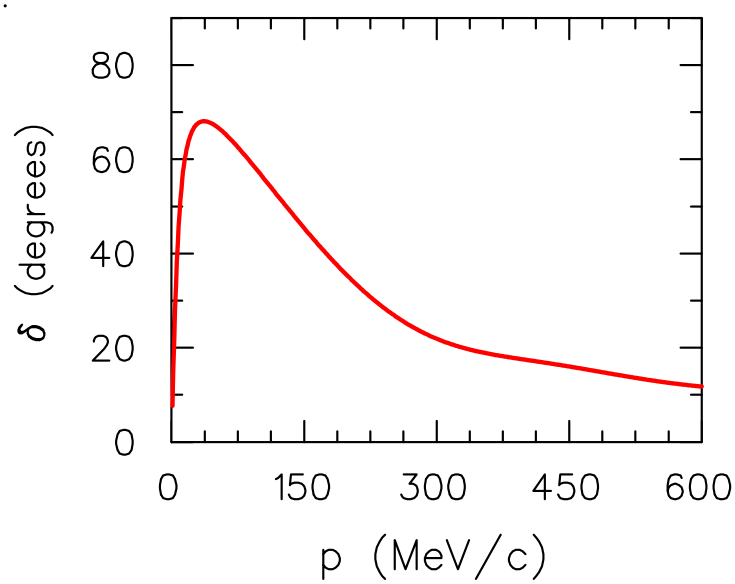
$$V(x) = \begin{cases} \infty, & x < 0 \\ -V_0\left(1 - \frac{r^2}{R^2}\right), & 0 < x < R \\ 0, & r > R \end{cases}$$

For this example, we will consider $R = 2.5$ fm, and V_0=16 MeV. If you wish, to make the units more natural, you may consider $\hbar c$=197.327 MeV·fm, and $m_p = 938.27$ MeV/c^2. Consider a particle incident on the well with energy E that enters and leaves the well with energy E. Far away, the solutions are of the form,

$$\psi(x) = e^{-ipx/\hbar} - e^{2i\delta + ipx/\hbar}\ , \quad x >> R$$

(a) Programming in either PYTHON or C++, construct a program that runs and returns a listing of δ vs. p for $0 < p < 600$ MeV/c, in steps of 2.0 MeV/c.

A graph of the results:



(b) **EXTRA CREDIT Make a graph like the one above, except for the region between p=0 and p=1.0 MeV, and consider two strengths of the potential, $V_0 = 17.0$ MeV and $V_0 = 17.025$ MeV. Be sure to calculate values for very small values of p, in steps of .001 MeV. For this problem, turn in a paper copy of the graph.**

Solution:

```
#include <cstdlib>
#include <cmath>
#include <cstdio>
#include <complex>
#include <string>
#include <cstring>

const double PI=4.0*atan(1.0);
const double HBARC=197.3269602;

using namespace std;

double V(double V0,double r){
   const double R=2.5;
   if(r>R)
      return 0.0;
   else
      return -V0*(1.0-r*r/(R*R));
}

double GetDelta(double V0,double p){
   const int NMAX=3000;
   const double Rmax=3.0;
   int n;
   complex<double> psi[NMAX+1],ci(0.0,1.0);
   double mu=938.27,C1,C2,r,q,delta,delr=Rmax/double(NMAX);
   q=p/HBARC;
   C1=q*q*delr*delr;
   C2=2.0*mu*delr*delr/(HBARC*HBARC);
   r=NMAX*delr; psi[NMAX]=exp(-ci*q*r);
   r=(NMAX-1)*delr; psi[NMAX-1]=exp(-ci*q*r);
   for(n=NMAX-2;n>=0;n--){
      r=(n+1)*delr;
      psi[n]=2.0*psi[n+1]-psi[n+2]+(-C1+C2*V(V0,r))*psi[n+1];
   }
   delta=-real(0.5*ci*log(psi[0]/conj(psi[0])));
   return delta;
}

int main(int argc,char *argv[]){
   double V0,p,delp=0.05,delta;
   printf("Enter V0: ");
   scanf("%lf",&V0);
   for(p=delp;p<10;p+=delp){
      delta=GetDelta(V0,p);
      if(delta<0.0)
         delta+=PI;
      printf("p=%6.2f delta=%g\n",p,delta*180.0/PI);
   }

   return 0;
}
```

5. Consider a potential which gives non-zero phase shifts for $0 \le \ell \le \ell_{\max}$, where $\ell_{\max}$ is a large number. Assume these phase shifts can be considered as random numbers, evenly distributed between zero and 2π. Using the expression for the cross section,

$$\sigma = \frac{4\pi\hbar^2}{p^2} \sum_\ell (2\ell+1)\sin^2\delta_\ell,$$

   (a) Find the overall cross section by averaging over the expectation of the random phases. Give your answer in terms of $\ell_{\max}$ and the incoming momentum p.

   (b) Consider a problem classically where one scatters off a strong central potential whose maximum range is $R_{\max}$. From classical arguments, what is the maximum angular momentum of a particle that scatters? Give your answer in terms of $R_{\max}$ and the incoming momentum p. What is the total cross section in terms of $R_{\max}$ in the limit that $\ell_{\max}$ is large.

Solution:
For random phase shifts the average of $\sin^2\delta$ is $1/2$.

$$\begin{aligned}
\sigma &= \frac{4\pi\hbar^2}{2mp^2} \sum_{\ell=0}^{\ell_{\max}} (2\ell+1)\frac{1}{2} \\
&\approx \frac{4\pi\hbar^2}{p^2} \sum_{\ell=0}^{\ell_{\max}} \ell \\
&\approx \frac{4\pi\hbar^2}{p^2} \frac{\ell_{\max}^2}{2} \\
&= \frac{2\pi\hbar^2}{p^2} \ell_{\max}^2.
\end{aligned}$$

Now substitute

$$\hbar\ell_{\max} = pR_{\max},$$

So

$$\sigma = 2\pi R_{\max}^2.$$

Classically,

$$\sigma_{\text{classical}} = \pi R_{\max}^2.$$

Thus, it is twice the geometric cross section. This doubling is due to diffraction.

6. A particle of mass m experiences an attractive spherically symmetric potential,

$$V(r) = -\beta\delta(r-a),$$

where $\beta > 0$.

(a) In terms of a, and the electron mass m, what is the minimum value of β that results in a bound state?

(b) What is the scattering length and the cross section in the limit that the incident beam energy is zero.

(c) If a scattered wave in a large volume behaves as

$$\psi(\vec{k},\vec{r},t) \sim e^{i\vec{k}\cdot\vec{r}-i\omega t},\ t \to \infty$$

in the outgoing limit (large time after interacting with potential), what is the relative probability,

$$\alpha(k) = \frac{\rho(\vec{r}=0)}{\rho_0(\vec{r}=0)},$$

that it will appear at the origin while interacting with the potential? Here ρ_0 is the probability density (per unit volume) in the absence of the potential, and ρ is the probability density with the potential in place. FYI: The ratio α would be the same if the boundary conditions specified an incoming plane wave, instead of matching to an outgoing plane wave.

(d) Assume β is sufficiently large to bind a particle, and that the ground state energy is $-B$. For the ground state what is the probability density of finding the particle at $\vec{r}=0$? Refer to this as $\rho_b(\vec{r}=0)$? Given answer in terms of a and the binding energy B (or equivalently the decay wave number, $q \equiv \sqrt{2mB/\hbar^2}$). HINT: You don't need to solve for the binding energy!

Solution:
a)

$$\begin{aligned}\psi_I &= A\sinh(qr),\\ \psi_{II} &= e^{-qr},\\ \text{B.C.}:\ A\sin(qa) &= e^{-qa},\\ aA\cosh(qa) + qe^{-qa} &= \frac{2m\beta}{\hbar^2}e^{-qa}.\end{aligned}$$

Eliminate A,

$$\tanh(qa) = \frac{1}{[2m\beta/(\hbar^2 q)]-1} = q\frac{\hbar^2/(2m\beta)}{1-\hbar^2 q/(2m\beta)}.$$

Both the tanh and the r.h.s. functions begin as linear function. The tanh function bends down with increasing a while the r.h.s. bends upwards. If the two are to intersect (and have a bound state solution), the slope of the tanh function must start lower. Thus for a bound state

$$a < \frac{\hbar^2}{(2m\beta)},$$
$$\beta > \frac{\hbar^2}{2ma}.$$

b) Consider $k \to 0$, and because $\delta(k=0) = 0$ you can write $\delta(k) \approx k d\delta/dk$.

$$\begin{aligned}
\psi_I = \sin(kr), \quad \psi_{II} &= \sin(kr+\delta), \\
A\sin(ka) &= \sin(ka+\delta), \\
kA\cos(ka) - k\cos(ka+\delta) &= \frac{2m\beta}{\hbar^2} A\sin(ka), \\
\frac{\sin(ka)}{k\cos(ka) - (2m\beta/\hbar^2)\sin(ka)} &= \frac{\sin(ka+\delta)}{k\cos(ka+\delta)}, (1) \\
\text{As } k \to 0,\ \frac{a}{1-2m\beta a/\hbar^2} &= a + \frac{d\delta}{dk}, \\
\frac{d\delta}{dk} &= -a + \frac{a}{1-2m\beta a/\hbar^2} = \frac{2m\beta a^2\hbar^2}{1-2m\beta a/\hbar^2}, \\
\text{scatt. length } \ell &= -\frac{d\delta}{dk} = -\frac{2m\beta a^2\hbar^2}{1-2m\beta a/\hbar^2}
\end{aligned}$$

As $k \to 0$,

$$\begin{aligned}
\sigma &= \frac{4\pi}{k^2}\sin^2\delta = 4\pi\left(\frac{d\delta}{dk}\right)^2 \\
&= 4\pi a^2 \left(\frac{2m\beta a\hbar^2}{1-2m\beta a/\hbar^2}\right)^2.
\end{aligned}$$

c) At small r, $u_\ell(r) \approx Akre^{i\delta}$, and the wave function $R_\ell(r \to 0) = u_\ell/(kr) = Ae^{i\delta}$. Looking at the partial wave expansion, and realizing that only the $\ell = 0$ term contributes at $r = 0$, one can see that the wave function at $r = 0$ is

$$\begin{aligned}
\psi(r=0) &= R_{\ell=0}(r=0)/\sqrt{V} = A/\sqrt{V}, \\
|\psi(r=0)|^2 &= \frac{A^2}{V}.
\end{aligned}$$

In the absence of the potential the wave function would be $e^{i\vec{k}\cdot\vec{r}}/\sqrt{V}$ and density would be $1/V$. So the interaction enhances the density at the origin by a factor of A^2.
Solving for A from the BC above,

$$\begin{aligned}
A^2 &= \frac{\sin^2(ka+\delta)}{\sin^2(ka)}, \\
&= \frac{\tan^2(ka+\delta)}{1+\tan^2(ka+\delta)} \frac{1}{\sin^2(ka)}.
\end{aligned}$$

Using the previous expression (1),

$$\tan(ka+\delta) = \frac{\sin(ka)}{\cos(ka) - [2m\beta/(\hbar^2 k)]\sin(ka)}.$$

Plugging this in and rearranging,

$$\begin{aligned}\alpha = A^2 &= \frac{1}{(\cos(ka) - [2m\beta/(\hbar^2 k)]\sin(ka))^2} \frac{1}{1 + \frac{\sin^2(ka)}{(\cos(ka) - [2m\beta/(\hbar^2 k)]\sin(ka))^2}} \\ &= \frac{1}{(\cos(ka) - [2m\beta/(\hbar^2 k)]\sin(ka))^2 + \sin^2(ka)}.\end{aligned}$$

Note that for $\beta = 0$ you indeed get $\alpha = 1$ as expected.

7. Near a resonance of energy ϵ_R, a phase shift behaves as:

$$\tan\delta_\ell = \frac{\Gamma/2}{\epsilon_R - E},$$

where E is the c.m. kinetic energy. For the following problems, assume that $\Gamma << \epsilon_R$, so that the $4\pi/k^2$ prefactor in the expression for the cross section can be considered as a constant.

(a) Write down the cross section $\sigma_\ell(E)$.

(b) What is the maximum cross section for a narrow cross section (as E is varied) for scattering through that partial wave? (How does it depend on ϵ_R, Γ, the reduced mass μ, and ℓ)?

(c) What is the energy integrated cross section ($\int \sigma_\ell(E)dE$)?

Solution:
a)

$$\sigma = \tfrac{4\pi}{k^2}\sin^2\delta \tag{0.1}$$

$$= \tfrac{4\pi}{k^2}\left(1 - \tfrac{1}{1+\tan^2\delta}\right) \tag{0.2}$$

$$= \tfrac{4\pi}{k^2}\left(\tfrac{\tan^2\delta}{1+\tan^2\delta}\right) \tag{0.3}$$

$$\tfrac{4\pi}{k^2}\tfrac{(\Gamma/2)^2/(\epsilon_R-E)^2}{1+(\Gamma/2)^2/(\epsilon_R-E)^2} \tag{0.4}$$

$$= \tfrac{4\pi}{k^2}\tfrac{(\Gamma/2)^2}{(\Gamma/2)^2+(\epsilon_R-E)^2}, \tag{0.5}$$

$$\tag{0.6}$$

b) The maximum cross section is

$$\sigma_{\max} = \frac{4\pi}{k_R^2},$$

$$\frac{\hbar^2 k_R^2}{2\mu} = \epsilon_R,$$

$$k_R^2 = \frac{2m\epsilon_R}{\hbar^2}.$$

c) Approximate the $1/k^2$ as $1/k_R^2$ for a narrow resonance.

$$\int dE\sigma(E) = \tfrac{4\pi}{k^2}\int dE \tfrac{(\Gamma/2)^2}{(\Gamma/2)^2+(\epsilon_R-E)^2}. \tag{0.7}$$

Substitute $\tan\theta = (\epsilon_R - E)/(\Gamma/2)$, then

$$\int dE\sigma(E) = \frac{4\pi}{k^2}\frac{\pi\Gamma}{2}.$$

Thus, narrower resonances integrate to smaller values because the range of their influence is proportional to Γ and the maximum cross section depends only on the wave number for resonance, k_R.

8. The temperature at the center of the sun is 15 million degrees Kelvin. Consider two protons with a relative kinetic energy characteristic of the temperature,

$$\frac{\hbar^2 k^2}{2\mu} = \frac{3}{2}kT.$$

(a) What is the Gamow penetrability factor? Give a numeric value.

(b) If the two particles were a proton and a ^{12}C nucleus, what would the penetrability factor become?

Solution:
a)

$$G = \frac{2\pi\gamma}{e^{2\pi\gamma} - 1}, \quad \gamma = \frac{1}{ka_0},$$
$$T = 15 \times 10^6 K = \frac{15 \times 10^6 \text{ K}}{1.1605 \times 10^4 \text{ K/eV}} = 1.3keV. \frac{\hbar^2 k^2}{2\mu} = \frac{3}{2}1.3keV,$$
$$k = \sqrt{\frac{3\mu\ 1.3keV}{\hbar}}, \quad \hbar c = 197.327\text{eV nm}, \hbar ck = 1.91 \text{ MeV},$$
$$\gamma = \frac{m_p c^2}{2} \frac{1}{137.036} \frac{1}{\hbar ck} = 1.79,$$
$$G = 1.44 \times 10^{-4}.$$

b)

$$\gamma \equiv \frac{\mu Z_1 Z_2 e^2}{\hbar^2 k},$$
$$\gamma = \frac{6m_p}{137.036} \frac{1.91}{\sqrt{2}} = 55,$$
$$G = 1.25 \times 10^{-149}.$$

.

9. Consider a particle of mass m undergoing a repulsive spherically symmetric Coulomb potential, $V = Ze^2/r$. The classical analogue of the squared wave function is

$$|\phi(\vec{p}_f, \vec{r})|^2 \to \frac{d^3p_i}{d^3p_f}.$$

Here, $\vec{p}_i$ is the momentum when the particle is at position $\vec{r}$, and $\vec{p}_f$ is the asymptotic momentum at large times.

(a) If one averages over all directions of the final momentum, what is $\langle|\phi(\vec{p}_f, \vec{r})|^2\rangle$? Give a sketch of the classical approximationn to $\langle|\phi(\vec{p}_f, \vec{r})|\rangle$ as a function of r for fixed p.

(b) Repeats (a) but working in two dimensions, i.e. find d^2p_i/d^2p_f.

(c) Repeats (a) and (b) but working in one dimension, i.e. find dp_i/dp_f.

Solution:
a)

$$\begin{aligned}
\frac{p_i^2}{2m} + \frac{Ze^2}{r} &= \frac{p_f^2}{2m}, \\
p_i dp_i &= p_f dp_f, \\
\left|\frac{d^3p_i}{d^3p_f}\right| &= \frac{p_i}{p_f}\frac{p_i dp_i}{p_f dp_f} \\
&= \frac{p_i}{p_f} \\
&= \sqrt{\frac{p_f^2 - 2me^2/r}{p_f^2}}.
\end{aligned}$$

However, there is no p_i for $p_f^2/2m < Ze^2/r$, so the value is zero for small r.

$$|\phi(\vec{p}_f, \vec{r})|^2 \to \begin{cases} \sqrt{\frac{p_f^2 - 2me^2/r}{p_f^2}}, & r > 2mZe^2/p_f^2 \\ 0, & r < 2mZe^2/p_f^2 \end{cases}$$

b) In two dimensions

$$|\phi(\vec{p}_f, \vec{r})|^2 \to \frac{p_i dp_i}{p_f dp_f} = 1,$$

when energetically allowed.

$$|\phi(\vec{p}_f, \vec{r})|^2 \to \begin{cases} 1, & r > 2mZe^2/p_f^2 \\ 0, & r < 2mZe^2/p_f^2 \end{cases}$$

In one dimension b) In two dimensions

$$|\phi(\vec{p}_f, \vec{r})|^2 \to \frac{p_i dp_i}{p_f dp_f} = \frac{p_f}{p_i},$$

when energetically allowed.

$$|\phi(\vec{p}_f,\vec{r})|^2 \to \begin{cases} \sqrt{\frac{p_f^2}{p_f^2-2me^2/r}}, & r > 2mZe^2/p_f^2 \\ 0, & r < 2mZe^2/p_f^2 \end{cases}$$