Chapter 6 – Homework Solutions

1. Using the WKB approximation, you can estimate the lifetime of a particle of mass m initially trapped in the "ground state" of a one-dimensional rectangular well using Eq. (**??**),

$$V(x) = \begin{cases} \infty, & x < 0 \\ 0, & 0 < x < a \\ \frac{\alpha}{x}, & a < x \end{cases}$$

Assume the barrier is sufficiently high that the wave function in the well can be approximated as that of an infinite well and that the frequency of tunneling attempts can be thought of as the rate at which a classical particle would impact the barrier at that energy.

   (a) Estimate the energy E, by treating the potential as if its infinity at $x = a$.

   (b) To what point, x_f, does the particle need to tunnel to escape the well.

   (c) Estimate the tunneling probability, $e^{-2\phi}$, where ϕ is calculated using the WKB approximation. I.e. find an expression for ϕ. Warning: The integral is not trivial.

   (d) In terms of ϕ what is the lifetime?

Solution:

a)

$$\begin{aligned}
\text{Probability} &\approx e^{-2\phi}, \\
\phi &= \int_a^{x_f} dx\ \sqrt{2m(V(x)-E)/\hbar^2}, \quad x_f = \alpha/E. \\
\phi &= \sqrt{\frac{2mE}{\hbar^2}} \int_a^{x_f} dx\ \sqrt{(x_f/(x)-1)}, \\
\text{sub } u &= x_f/x,\ du = -dx x_f/x^2,\ dx = -(du/u^2)x_f, \\
\phi &= \sqrt{2mx_f^2 E/(\hbar^2)} I(x_f/a), \\
I(y) &\equiv \int_1^y \frac{du}{u^2}\sqrt{u-1}, \\
\text{sub } z &= \sqrt{u-1},\ u-1 = z^2,\ du = 2zdz, \\
I(y) &= 2\int_0^{\sqrt{y-1}} dz \frac{z^2}{(1+z^2)^2}. \\
\text{sub } z &= \tan\theta, dz/(1+z^2) = d\theta, \\
I(y) &= 2\int_0^{\tan^{-1}(\sqrt{y-1})} d\theta \frac{\tan^2\theta}{1+\tan^2\theta} \\
&= 2\int_0^{\tan^{-1}(\sqrt{y-1})} d\theta\ \sin^2\theta \\
&= \int_0^{\tan^{-1}(\sqrt{y-1})} d\theta(1-\cos 2\theta) \\
&= \left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\tan^{-1}(\sqrt{y-1})}, \\
\frac{1}{2}\sin(2\theta) &= \sin\theta\cos\theta = \sqrt{1-\cos^2\theta}\cos\theta = \sqrt{1-\frac{1}{1+\tan^2\theta}}\sqrt{\frac{1}{1+\tan^2\theta}}. \\
I(y) &= \tan^{-1}(\sqrt{y-1}) - \sqrt{\frac{y-1}{y}}\sqrt{\frac{1}{y}} \\
&= \tan^{-1}(\sqrt{y-1}) - \frac{1}{y}\sqrt{y-1}, \\
\phi &= \sqrt{2mx_f^2 E/\hbar^2} I(x_f/a), \quad x_f = \alpha/E.
\end{aligned}$$

b) The tunneling success probability is approximately $e^{-2\phi}$ and the trial rate is $v/(2a)$, where the "velocity" is given the classical value,

$$v = \sqrt{2E/m}.$$

The lifetime is then

$$\tau \approx \sqrt{\frac{2ma^2}{E}} e^{2\phi}.$$

where ϕ is given above.

2. A particle of mass m is initially in the ground state of a one-dimensional harmonic oscillator of frequency ω centered at $x = 0$. Suddenly, at time $t = 0$, the center of the well is moved to $x = \ell$. Here, you will calculate the probability that the particle will be found in the state $|n\rangle$ of the new well, where $|0\rangle$ is the new ground state.

   (a) The ground state of the old well can be written as

   $$|\phi_0\rangle = e^{\ell(d/dx)}|0\rangle,$$

   where ϕ_0 is the ground state of the old well and $|n\rangle$ refers to eigenstates of the newly positioned well. Writing the operator d/dx in terms of creation and destruction operators of the new well, find an expression for $|\phi_0\rangle$ as a linear combination of $|n\rangle$. HINT: You will probably wish to use the Baker-Campbell-Hausdorff relation.

   (b) What is the probability of finding the particle in the state $|n\rangle$.

   (c) What is the expectation of the energy $\langle H\rangle$ after the well is shifted?

   (d) If the well were shifted slowly instead of suddenly to its new position, what would be the probability of finding the particle in the ground state of the new well?

Solution:
a)

$$\begin{aligned}
|\phi_0\rangle &= e^{\ell\partial_x}|0\rangle,\\
\partial_x &= \frac{ip}{\hbar} = \frac{1}{\hbar}\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger) = \frac{1}{2^{1/2}b}(a - a^\dagger), \qquad b^2 = \frac{\hbar}{m\omega_0},\\
e^{\ell/(2^{1/2}b)(a-a^\dagger)} &= e^{-\ell/(2^{1/2}b)a^\dagger}e^{\ell/(2^{1/2}b)a}e^{-(1/2)(\ell^2/2b^2)[a,a^\dagger]}
\end{aligned}$$

The far-right exponential is just a number as $[a, a^\dagger] = 1$. The center exponential can be expanded in a Taylor expansion, so when operating on the vacuum, $|0\rangle$, only the first term survives so it becomes unity. Thus,

$$e^{\ell/(2^{1/2}b)(a-a^\dagger)}|0\rangle = e^{-(1/2)(\ell^2/2b^2)}e^{-\ell/(2^{1/2}b)a^\dagger}|0\rangle.$$

The exponential with the $a^\dagger$ can also be expanded, so the overlap with the original ground state is thus

$$\langle 0|e^{\ell/(2^{1/2}b)(a-a^\dagger)}|0\rangle = e^{-(1/2)(\ell^2/2b^2)}.$$

The probability of staying in the ground state is the square of the matrix element,

$$\text{Prob} = e^{-(\ell^2/2b^2)}.$$

b) Take the overlap into the state $|n\rangle$

$$\langle n|e^{\ell\partial_x}|0\rangle = e^{-(1/2)(\ell^2/2b^2)}\langle n|e^{(-\ell)/(2^{1/2}b)a^\dagger}|0\rangle.$$

Expanding the exponential and using the fact that $(a^\dagger)^n|0\rangle = \sqrt{n!}|n\rangle$,

$$\begin{aligned}
\langle n|e^{\ell\partial_x}|0\rangle &= e^{-(1/2)(\ell^2/2b^2)}\frac{[-\ell/(2^{1/2}b)]^n}{\sqrt{n!}},\\
\text{Prob}(n) &= e^{-(\ell^2/2b^2)}\frac{[(\ell^2/(2b^2)]^n}{n!}
\end{aligned}$$

Note that if you sum over n you will get unity. c)

$$
\begin{aligned}
\langle H \rangle &= \sum_n \mathrm{Prob}(n)\hbar\omega_0(n+1/2) \\
&= e^{-(\ell^2/2b^2)}\hbar\omega_0 \sum_{n=0}^{\infty} \frac{[(\ell^2/(2b^2)]^n}{n!}(n+1/2).
\end{aligned}
$$

For the 1/2 in $(n+1/2)$, that simply sums to 1/2 by inspection.

$$
\langle H \rangle = e^{-(\ell^2/2b^2)}\hbar\omega_0/2 + e^{-(\ell^2/2b^2)}\hbar\omega_0 \sum_{n=0}^{\infty} \frac{[(\ell^2/(2b^2)]^n}{n!} n.
$$

By pulling out a factor of $(\ell^2/(2b^2)$ from the sum, you can see that the rest of the sum combined with the preceding exponential will be unity by replacing $n-1$ with n,

$$
\langle H \rangle = e^{-(\ell^2/2b^2)}\hbar\omega_0/2 + \hbar\omega_0 \frac{\ell^2}{2b^2} = \frac{\hbar\omega_0}{2} + \frac{1}{2}m\omega_0^2\ell^2.
$$

d) 100%.

3. Estimate the ground state energy of the hydrogen atom using a three-dimensional harmonic oscillator ground state wave function as a trial function. Give your answer in terms of the electron mass m and the coupling e^2.

Solution:

$$
\begin{aligned}
\psi &= \frac{1}{(\pi b^2)^{3/4}} e^{-r^2/2b^2}, \\
\langle KE \rangle &= 3 \left\langle \frac{p_x^2}{2m} \right\rangle \quad \text{(by symmetry)} \\
&= 3\frac{\hbar^2}{2m} \int \frac{dx}{(\pi b^2)^{1/2}} e^{-x^2/2b^2} \partial_x^2 e^{-x^2/2b^2} \\
&= -3\frac{\hbar^2}{2m} \int \frac{dx}{(\pi b^2)^{1/2}} e^{-x^2/2b^2} \left[\frac{x^2}{b^4} - \frac{1}{b^2} \right] e^{-x^2/2b^2} \\
&= 3\frac{\hbar^2}{2m}\frac{1}{2b^2} = \frac{3\hbar^2}{4mb^2}.
\end{aligned}
$$

Alternatively, you could have used the equipartition theorem for the harmonic oscillator and written down the answer.

$$
\begin{aligned}
\langle V \rangle &= -\int \frac{4\pi r^2 dr}{(\pi b^2)^{3/2}} e^{-r^2/b^2} \frac{e^2}{r} \\
&= \frac{-2\imath e^2}{(\pi b^2)^{3/2}} \int du \; e^{-u^2/b^2} \\
&= -\frac{2e^2}{\pi^{1/2} b}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{db}\left(\frac{3\hbar^2}{4m}\frac{1}{b^2} - \frac{2e^2}{\pi^{1/2} b} \right) &= 0, \\
\frac{3\hbar^2}{2m}\frac{1}{b^3} &= \frac{2e^2}{\pi^{1/2} b^2}, \\
b &= \frac{3\hbar^2 \pi^{1/2}}{4me^2}.
\end{aligned}
$$

Plug this expression for b into the expressions for V and KE to get (after some algebra)

$$
\begin{aligned}
E &= \langle KE \rangle + \langle V \rangle \\
&= \frac{4me^4}{3\hbar^2 \pi} - \frac{8me^4}{3\pi\hbar^2} \\
&= -\frac{4}{3}\frac{me^4}{\pi\hbar^2}.
\end{aligned}
$$

This is a factor of $8/(3\pi)$ smaller than the true binding energy.

4. Estimate the ground state energy of the three-dimensional harmonic oscillator using the hydrogen atom wave function as the trial wave function. Give your answer in terms of the mass m and the characteristic frequency of the harmonic oscillator ω.

Solution:

$$
\begin{aligned}
\psi(r) &= e^{-r/b}\frac{1}{\pi^{1/2}b^{3/2}},\\
\langle KE\rangle &= -\frac{\hbar^2}{2m}4\pi\int r^2dr\ \frac{1}{\pi b^3}e^{-r/b}\left(\partial_r^2+\frac{2}{r}\partial_r\right)e^{-r/b}\\
&= -\frac{2\hbar^2}{mb^3}\int r^2dr\ e^{-2r/b}\left(-\frac{1}{b^2}+\frac{2}{br}\right)\\
&= -\frac{2\hbar^2}{mb^2}\left[-2(1/2)^3+2(1/2)^2\right]\\
&= \frac{\hbar^2}{2mb^2}.\\
\langle V\rangle &= 4\pi\int r^2dr\ \frac{1}{\pi b^3}e^{-2r/b}\frac{1}{2}kr^2,\\
&= \frac{2k}{b^3}\int f^4dr!e^{-2r/b}\\
&= \frac{3}{2}kb^2\\
\frac{d}{db}\left\{\frac{\hbar^2}{2mb^2}+\frac{3}{2}kb^2\right\} &= \frac{\hbar^2}{mb^3}+3kb=0,\\
b^2 &= \left(\frac{\hbar^2}{3mk}\right)^{1/2},
\end{aligned}
$$

Plugging the expression for b into the expressions for $\langle KE\rangle$ and $\langle V\rangle$,

$$E=\sqrt{3}\hbar\omega_0.$$

This is larger that the true ground state energy by a factor $2\sqrt{3}/3$.

5. Consider a particle in an infinitely deep square well of width a.

$$V_0(x) = \begin{cases} \infty, & x < -a/2 \\ 0, & -a/2 < x < a/2 \\ \infty, & x > a/2 \end{cases}$$

A particle feels a perturbative potential,

$$V_1(x) = \beta \sin(\pi x/a)$$

(a) What is the change in the ground state energy in lowest (non-zero) order perturbation theory?

(b) What is the correction to the energy of the first excited state to the same order?

(c) What is the correction to the wave function of the ground state to lowest non-zero order?

Solution:
a) $\Delta E^{(1)} = 0$ because the perturbation is odd.
b) Also zero, same reason
c)

$$\begin{aligned}
\epsilon_n &= \frac{\hbar^2}{2m} k_n^2, \quad k_n a = (n+1)\pi, \\
\epsilon_n &= \frac{\hbar^2 \pi^2}{2m}(n+1)^2, \\
\Delta E^{(2)} &= -\sum_{n=\text{odd}}^{\infty} \frac{|\langle n|V|0\rangle|^2}{\epsilon_n - \epsilon_0} \\
\psi_{n=\text{odd}}(x) &= \sqrt{\frac{2}{a}} \sin(2m\pi x/a), \quad m = (n+1)/2 \\
&= \sqrt{\frac{2}{a}} \sin((n+1)\pi x/a), \\
\psi_0(x) &= \sqrt{\frac{2}{a}} \cos(\pi x/a), \\
\langle n = \text{odd}|V|0\rangle &= \frac{2\beta}{a} \int_{-a/2}^{a/2} dx \; \sin[(n+1)\pi x/a] \cos(\pi x/a) \sin(\pi x/a), \\
&= \frac{\beta}{a} \int_{-a/2}^{a/2} dx \; \sin[(n+1)\pi x/a] \sin(2\pi x/a).
\end{aligned}$$

The last sin term is a solution, and by orthogonality, one can see that only the $n = 1$ term contributes from the sum.

$$\langle n = \text{odd}|V|0\rangle = \frac{\beta}{2}\delta_{n,1}.$$

The energy difference between the $n = 0$ and $n = 1$ states is

$$\epsilon_1 - \epsilon_0 = \frac{3\hbar^2\pi^2}{2ma^2},$$

so

$$\Delta E^{(2)} = -\frac{ma^2\beta^2}{6\hbar^2\pi^2}.$$

6. Consider the Hamiltonian:

$$H_0 = \alpha\sigma_z$$

and the perturbation

$$V = \beta\sigma_x$$

(a) What is the correction to the ground state energy to second order in perturbation theory?

(b) What is the correction to the excited state's energy to the same order?

(c) Find the exact expression for the energy of the first state, and show that it gives the same answer as part a when expanded in powers of β.

Solution:

a)

$$\epsilon_0 = -\alpha, \quad \epsilon_1 = \alpha.$$

The first-order correction is zero because $\langle 0|\sigma_x|0\rangle = 0$.

$$\begin{aligned}\Delta E^{(2)} &= -\frac{\beta^2}{2\alpha}|\langle 1|\sigma_x|0\rangle|^2 \\ &= -\frac{\beta^2}{2\alpha}.\end{aligned}$$

b) Same matrix, but opposite sign in the denominator,
$\frac{\beta^2}{2\alpha}$.

c)

$$\begin{aligned}H &= \sqrt{\alpha^2+\beta^2}\vec{\sigma}\cdot\hat{n}, \\ E &= \pm\sqrt{\alpha^2+\beta^2}, \\ &= \pm\alpha\sqrt{1+\beta^2/\alpha^2} \\ &\approx \pm(\alpha+\frac{\beta^2}{2\alpha}).\end{aligned}$$

7. An electron of mass m initially in the ground state of a three-dimensional harmonic oscillator potential characterized by frequency ω, i.e. the ground state energy is $3\hbar\omega/2$, is placed in a region with uniform electric field E.

   (a) By finding corrections to the ground state wave function in first order perturbation theory, find an expression for the electric dipole moment induced in the atom.

   (b) An alternative method for calculating the dipole moment is to differentiate the energy with respect to the electric field. Show that this method yields the same expression found in (a) when one uses second order perturbation theory to find the correction to the energy.

Solution:

$$\begin{aligned}
V &= -eEx = eE(a + a^\dagger)\sqrt{\frac{\hbar}{2m\omega_0}},\\
|\psi\rangle &= |\psi_0\rangle + \sum_m \frac{\langle 0|V|m\rangle}{\epsilon_m - \epsilon_0}|\psi_m\rangle
\end{aligned}$$

Only the $m = 1$ term contributes

$$\begin{aligned}
|\psi\rangle &= |\psi_0\rangle + \frac{eE\sqrt{\hbar/(2m\omega)}}{\hbar\omega}|\psi_1\rangle\\
&= |\psi_0\rangle + \frac{eE}{\sqrt{2\hbar m\omega_0^3}}|\psi_1\rangle,\\
p &= -\langle\psi|ex|\psi\rangle\\
&= -\frac{e^2E}{\sqrt{2\hbar m\omega_0^3}}\langle 0|x|1\rangle + h.c.
\end{aligned}$$

Using the expression for x in terms of $(a + a^\dagger)$ above,

$$\begin{aligned}
p &= -2\frac{e^2E}{\sqrt{\hbar m\omega_0^3}}\sqrt{\frac{\hbar}{2m\omega_0}}\\
&= \frac{e^2E}{m\omega_0^2}.
\end{aligned}$$

8. Two electrons whose positions are defined by $\mathbf{r}_1$ and $\mathbf{r}_2$ relative to the centers of their confining potentials. The confining potentials are then separated by a distance $\vec{R}$.

$$V_0(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2} m\omega^2 (r_1^2 + r_2^2).$$

Positive charges $+e$ are fixed at the centers of the potentials. The electromagnetic energy between the two wells is:

$$V = \frac{e^2}{R} + \frac{e^2}{|\vec{R} + \vec{r}_1 - \vec{r}_2|} - \frac{e^2}{|\vec{R} + \vec{r}_1|} - \frac{e^2}{|\vec{R} - \vec{r}_2|}.$$

Here, the repulsive interaction between the two positive ions is described by the first term, and the repulsive interaction between the electrons is described by the second term. The last two terms describe the attractive interaction between the electron and the ion in the other well. The electromagnetic energy between each electron and its confining ion is assumed to be part of the confining potential, and not part of the perturbation.

Assume that the separation R is much larger than either r_1 or r_2. In terms of the separation between the wells, R, the mass of the electrons m, the charge e and ω,

(a) Show that for large R, the interaction may be approximated as a dipole-dipole interaction,

$$V = \frac{e^2}{R^3} \left(x_1 x_2 + y_1 y_2 - 2 z_1 z_2 \right),$$

where the z axis is along the direction of $\vec{R}$.

(b) Use second-order perturbation theory to find the electromagnetic attraction of the two wells, $V(R)$. This motivates the form for the London dispersion force, https://en.wikipedia.org/wiki/London_dispersion_force, which is the long-range attractive force between neutral molecules.

Solution:
a) First, some calculus for expanding $1/|\vec{R} + \vec{a}|$, and keeping terms of to order a^2

$$\begin{aligned}
|\vec{R} + \vec{a}| &= \sqrt{R^2 + 2\vec{a} \cdot \vec{R} + a^2} \\
&= R\sqrt{1 + 2\vec{a} \cdot \vec{R}/R^2 + a^2/R^2}, \\
\frac{1}{|\vec{R} + \vec{a}|} &\approx \frac{1}{R} \left\{ 1 - \frac{1}{2} \left(2\vec{a} \cdot \vec{R}/R^2 + a^2/R^2 \right) + \frac{3}{8} \left(2\vec{a} \cdot \vec{R}/R^2 \right)^2 \right\} \\
&= \frac{1}{R} - \frac{\vec{a} \cdot \vec{R}}{R^3} - \frac{a^2}{2R^3} + \frac{3}{2} \frac{(\vec{a} \cdot \vec{R})^2}{R^5}.
\end{aligned}$$

Next, write the inter-atom interaction, using the shorthand

$$U(R, \vec{a}) \equiv \frac{e^2}{|R\hat{z} + \vec{a}|}$$

$$V = U(R, 0) - U(R, \vec{r}_1) - U(R, -\vec{r}_2) + U(R, \vec{r}_1 - \vec{r}_2).$$

Using the expansion above, one can see that all the terms of order $1/R$ or r/R^2 cancel. This gives

$$
\begin{aligned}
V &= \frac{e^2}{R^3}\left\{-\frac{1}{2}|\vec{r}_1-\vec{r}_2|^2+\frac{1}{2}|\vec{r}_1|^2+\frac{1}{2}|\vec{r}_2|^2 -\frac{3}{2}(\vec{r}_1\cdot\vec{R})^2/R^2-\frac{3}{2}(\vec{r}_2\cdot\vec{R})^2/R^2+\frac{3}{2}[(\vec{r}_1-\vec{r}_2)\cdot\vec{R}]^2/R^2.\right\}\\
&= \frac{e^2}{R^3}\left\{\vec{r}_1\cdot\vec{r}_2-(\vec{r}_1\cdot\vec{R})(\vec{r}_2\cdot\vec{R})/R^2\right\}\\
&= \frac{e^2}{R^3}\left\{x_1x_2+y_1y_2+z_1z_2-3z_1z_2\right\}\\
&= \frac{e^2}{R^3}\left\{x_1x_2+y_1y_2-2z_1z_2\right\}.
\end{aligned}
$$

b) Consider the x_1x_2 term by itself. Then multiply your answer by 6 to get the entire answer.

$$\Delta E^{(2)} = -6\frac{|\langle n_{1x}=1, n_{2x}=1|V|n_{1x}=0, n_{2x}=0\rangle|^2}{2\hbar\omega}.$$

The matrix element $\langle n_{1x}=1, n_{2x}=1|x_1x_2|n_{1x}=0, n_{2x}=0\rangle$ is simply the product of two matrix elements for one-dimensional oscillators,

$$\langle n_{1x}=1, n_{2x}=1|x_1x_2|n_{1x}=0, n_{2x}=0\rangle = \langle 1|x|0\rangle^2$$

$$\langle 1|x|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 1|(a+a^\dagger)|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}.$$

Putting everything together,

$$\Delta E^{(2)} = -\frac{3\hbar e^4}{2m^2\omega^3R^6}.$$

Note that this is the force between two atoms where for each atom the ground state has no dipole moment. Thus, this is an induced-dipole-induced-dipole interaction. If one or both atoms or molecules have dipole moment in their ground state, the force will be different, and fall off more slowly in R.

9. Consider the function

$$g(\omega) \equiv \mathrm{Im}\frac{1}{\omega - i\eta} = \frac{\eta}{\omega^2 + \eta^2}$$

where η is a positive real constant that approaches zero.

(a) What is $g(\omega = 0)$?

(b) What is $g(\omega \neq 0)$?

(c) Using trigonometric substitutions, evaluate

$$\int_{-\infty}^{\infty} d\omega\ g(\omega).$$

(d) Write an expression for a delta function in terms of $g(\omega)$.

Solution:
a) ∞
b) zero
c)

$$\begin{aligned}
\int d\omega\ g(\omega) &= \int d\eta \frac{\omega}{\omega^2 + \eta^2} \\
&\text{sub } x = \omega/\eta, \\
\int d\omega\ g(\omega) &= \int \frac{dx}{1 + x^2}, \\
&\text{sub } x = \tan\theta,\ dx = \sec^2\theta d\theta = (1 + \tan^2\theta) d\theta, \\
\int d\omega\ g(\omega) &= \int_{-\pi/2}^{\pi/2} d\theta = \pi.
\end{aligned}$$

d) Because the function is zero every where except at $\omega = 0$, is ∞ at $x = 0$, and integrates to π,

$$\frac{1}{\pi} g(\omega) = \delta(\omega).$$

10. A *bob* particle is in the ground state of a 3-dimensional harmonic oscillator characterized by a frequency ω,

$$V_0 = \frac{1}{2} m\omega^2 r^2$$

A perturbation is added that allows a *bob* particle to undergo a transformation into a *mary* particle. The *mary* particle does not feel the effects of the oscillator potential. The *bob* and *mary* particles have the same mass m. The perturbation is of the form,

$$V_{bm} = g\vec{\epsilon}_s \cdot \int d^3r \psi^*_{bob}(\mathbf{r}) \nabla \psi_{mary}(\mathbf{r}),$$

where ϵ_s is the unit polarization vector of the *mary* particle with polarization s, which refers to any of three directions.

(a) Find the total decay rate Γ (to any polarization). Hint: The answer will of the form

$$\Gamma = A\frac{mg^2b^3k^3}{\hbar^3},$$

where A is a dimensionless constant and b is the size of the harmonic oscillator wave function. You need to find A.

(b) Find the differential decay rate, $d\Gamma_s/d\cos\theta$, for a polarization $\vec{\epsilon}_s = \hat{z}$.

Solution:
a)

$$\begin{aligned}
\psi_m(\vec{r}) &= \frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{V}}, \\
\psi_b(\vec{r}) &= \frac{e^{-x^2/2b^2}}{(\pi b^2)^{3/4}}, \\
k &= \sqrt{2mE/\hbar^2}, \\
E &= \frac{3}{2}\hbar\omega, \\
V_{bm}(\epsilon, \vec{k}) &= \frac{ig\vec{\epsilon}\cdot\vec{k}}{b^{3/2}\pi^{3/4}V^{1/2}} I(k_x)I(k_y), I(k_z),
\end{aligned}$$

where $I(k)$ is given by the following integral, which can be performed by completing the square in the argument of the exponential,

$$\begin{aligned}
I(k) &= \int dx\ e^{ikx} e^{-x^2/2b^2} \\
&= e^{-k^2b^2/2} \int dx\ e^{-(x-ikb^2)^2/2b^2} \\
&= e^{-k^2b^2/2}\sqrt{2\pi b^2}.
\end{aligned}$$

The matrix element is now

$$V_{bm}(\epsilon,\vec{k}) = \frac{ig\vec{\epsilon}\cdot\vec{k}\pi^{3/4}(2b)^{3/2}}{V^{1/2}}e^{-|\vec{k}|^2b^2/2}.$$

From Fermi's golden rule, the rate to decay into a given state with momentum $\vec{k}$ and polarization $\vec{\epsilon}$ is

$$\Gamma(\vec{\epsilon},\vec{k}) = \frac{2\pi}{\hbar}\frac{g^2(\vec{\epsilon}\cdot\vec{k})^2\pi^{3/2}(2b)^3}{V}e^{-|\vec{k}|^2b^2}\delta(E_k - 3\hbar\omega/2).$$

Summing over all $\vec{k}$, the rate to decay into some polarization state $\vec{\epsilon}$,

$$\Gamma(\vec{\epsilon}) = \frac{V}{(2\pi)^3}\int 2\pi k^2 dk d\cos\theta_{k,\epsilon}\ \Gamma(\vec{\epsilon},\vec{k}),$$

where the angle is that between $\vec{k}$ and $\vec{\epsilon}$. So,

$$\Gamma(\vec{\epsilon}) = \frac{V}{(2\pi)^3}\frac{2\pi}{\hbar}\frac{g^2\pi^{3/2}(2b)^3}{V}\int 2\pi k^2 dk d\cos\theta\ k^2\cos^2\theta e^{-|\vec{k}|^2b^2}\delta(E_k - 3\hbar\omega/2),$$

Now, using the fact that

$$\delta(E_k - 3\hbar\omega/2) = \frac{m}{\hbar^2 k}\delta(k - \sqrt{2mE/\hbar^2}),$$

and rearranging factors (note that the volume cancels),

$$\begin{aligned}\Gamma(\vec{\epsilon}) &= \frac{V}{(2\pi)^3}\frac{2\pi}{\hbar}\frac{g^2\pi^{3/2}(2b)^3}{V}2\pi k^2\int d\cos\theta\ k^2\cos^2\theta e^{-|\vec{k}|^2b^2}\frac{m}{\hbar^2 k}\\ &= \frac{8mg^2b^3k^3\pi^{1/2}}{3\hbar^3}e^{-|\vec{k}|^2b^2}.\end{aligned}$$

All three polarizations give the same rate, so to get the total decay rate, multiply by three,

$$\Gamma = \frac{8\pi^{1/2}mg^2b^3k^3}{\hbar^3}e^{-|\vec{k}|^2b^2}.$$

b) For a given polarization, simple omit the integral over θ in the integral above, and one gets

$$\frac{d\Gamma_s}{d\cos\theta} = \frac{4\pi^{1/2}mg^2b^3k^3}{\hbar^3}\cos^2\theta e^{-|\vec{k}|^2b^2}.$$

The emission prefers to be parallel or anti-parallel to the polarization.