Chapter 3 – Homework Solutions

1. Using the equations of motion for the wave function, show that the density and current defined by

$$\rho(\vec{r},t) = e|\psi(\vec{r},t)|^2,$$
$$\vec{j}(\vec{r},t) = \frac{-ie\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2,$$

satisfies the continuity equation,

$$\partial_t\rho + \nabla\cdot\vec{j} = 0.$$

Solution:

$$\begin{aligned}
\frac{d\rho}{dt} &= \frac{i}{\hbar}(H\psi^*)\psi - \frac{i}{\hbar}\psi^* H\psi \\
&= \frac{i}{\hbar}\left\{\frac{\hbar^2}{2m}(\nabla^2\psi^*)\psi - \frac{\hbar^2}{2m}\psi^*(\nabla^2\psi)\right. \\
&\quad - \frac{\hbar e}{m}(\vec{A}\cdot\nabla\psi^*) - \frac{\hbar e}{m}\psi^*(\vec{A}\cdot\nabla\psi \\
&\quad \left. -\psi^*\frac{e^2|\vec{A}|^2}{2m}\psi + \left(\frac{e^2|\vec{A}|^2}{2m}\psi^*\right)\psi\ \right\}
\end{aligned}$$

The last two terms cancel. Thus,

$$\frac{d\rho}{dt} = \nabla\cdot\left\{\frac{(i\nabla\psi^*)\psi - \psi^*(i\nabla\psi)}{2m} - \frac{e\vec{A}}{m}\psi^*\psi\right\}.$$

The r.h.s. is $\nabla\cdot\vec{j}$.

2. Consider a particle of charge e traveling in the electromagnetic potentials

$$\mathbf{A}(\mathbf{r},t) = -\nabla\Lambda(\mathbf{r},t), \qquad \Phi(\mathbf{r},t) = \frac{1}{c}\frac{\partial\Lambda(\mathbf{r},t)}{\partial t}$$

where $\Lambda(\mathbf{r},t)$ is an arbitrary scalar function.

(a) What are the electromagnetic fields described by these potentials?

(b) Show that the wave function of the particle is given by

$$\psi(\mathbf{r},t) = \exp\left[-\frac{ie}{\hbar c}\Lambda(\mathbf{r},t)\right]\psi_0(\mathbf{r},t),$$

where ψ^0 solves the Schrödinger equation with $\Lambda = 0$.

(c) Let $V(\mathbf{r},t) = e\Phi(t)$ be a spatially uniform time varying potential. Show that

$$\psi(\mathbf{r},t) = \exp\left[-\frac{ie}{\hbar}\int_{-\infty}^{t}\Phi(t')dt'\right]\psi_0(\mathbf{r},t)$$

is a solution if ψ_0 is a solution with $\Phi = 0$.

Solution:
a)

$$\vec{E} = \frac{1}{c}\left(\nabla\partial_t\Lambda - \nabla\partial_t\Lambda\right) = 0 \ \checkmark$$
$$\vec{B} = \nabla\times(\nabla\Lambda), \text{ or } B_i = \epsilon_{ijk}\partial_j\partial_k\Lambda = 0 \ \checkmark.$$

b) We must show that $i\hbar\partial_t\psi = H\psi$.

$$i\hbar\partial_t\psi = \frac{e}{c}(\partial_t\Lambda)\psi + e^{-ie\Lambda/(\hbar c)}H_0\psi_0,$$
$$H\psi = \frac{(-i\hbar\nabla - e\vec{A}/c)^2}{2m}e^{-ie\Lambda/(\hbar c)}\psi_0 + \frac{e}{c}(\partial_t\Lambda)\psi,$$
$$(-i\hbar\nabla - e\vec{A}/c)e^{-ie\Lambda/(\hbar c)}\psi_0 = e^{-ie\Lambda/(\hbar c)}(-i\hbar\nabla\psi_0 - e\vec{A}/c - (e/c)\nabla\Lambda)\psi_0.$$

The last two terms cancel because $\vec{A} = -\nabla\Lambda$. One can then see verify $i\hbar\partial_t\psi = H\psi$.
c) Let

$$\Lambda(t) = c\int_{-\infty}^{\infty} dt'\ \Phi(t'),$$

From above,

$$\psi = e^{-ie\Lambda/(\hbar c)}\psi_0 = e^{-(ie/\hbar)\int_{-\infty}^{t} dt'\ \Phi(t')}\psi_0.$$

3. For a gauge transformation, described in Eq. (**??**), including the associated the phase change to the wave function ψ, described in Eq. (**??**),

   (a) Show that the charge density $e\psi^*\psi$ is unchanged by the gauge transformation
   (b) Show that the current

$$\vec{j} = \frac{e}{2m}\left[\psi^*(-i\hbar\nabla\psi) + (i\hbar\nabla\psi^*)\psi\right] - \frac{e}{mc}\vec{A}\psi^*\psi.$$

   is unchanged.
   (c) Show that $\langle\chi|H|\psi\rangle$ is unchanged in a gauge transformation where Λ is independent of time.

Solution:
a)

$$\psi = e^{-ie\Lambda/(\hbar c)}\psi_0, \quad \psi^*\psi = \psi_0^*\psi_0.$$

b)

$$\begin{aligned}\vec{j} &= \frac{e}{2m}e^{ie\Lambda/(\hbar c)}\psi_0^*(-i\hbar\nabla - 2e\vec{A}/c)\psi_0 + \frac{e}{2m}(i\hbar\nabla(e^{-ie\Lambda/(\hbar c)}\psi_0^*))\psi 0 e^{ie\Lambda/(\hbar c)}\\ &= \frac{e}{2m}\psi_0^*(-i\hbar\nabla\psi_0) + \frac{e}{2m}(i\hbar\nabla\psi_0^*)\psi_0 - \frac{e}{m}\psi_0^*\frac{e\vec{A}}{c}\psi_0 + \frac{e}{m}(i\hbar)\frac{ie}{\hbar c}(\nabla\Lambda)\psi_0^*\psi_0\end{aligned}$$

Let $\vec{A} = \vec{A}_0 - \nabla\Lambda$,

$$\begin{aligned}\vec{j} &= \frac{e}{2m}\left[\psi_0^*(-i\hbar\nabla - e\vec{A}_0/c)\psi + ((i\hbar\nabla - e\vec{A}_0/c)\psi_0^*)\psi_0\right]\\ &= \vec{j}_0 \quad \checkmark\end{aligned}$$

c)

$$\begin{aligned}\chi^* &= \chi_0 e^{-ie\Lambda/(\hbar c)},\\ \psi &= e^{-ie\Lambda/(\hbar c)}\psi_0,\\ H &= V(r) + (-i\hbar\nabla - e\vec{A}_0/c + e\nabla\Lambda/c)^2/2m,\\ H_0 &= V(r) + (-i\hbar\nabla - e\vec{A}_0/)^2/2m,\\ (-i\hbar\nabla - e\vec{A}_0/c + e\nabla\Lambda/c)\psi &= e^{-ie\Lambda/(\hbar c)}(-i\hbar\nabla - e\vec{A}_0/c)\psi_0.\end{aligned}$$

Thus,

$$\begin{aligned}\chi H\psi &= \chi e^{-ie\Lambda/(\hbar c)}H_0\psi_0\\ &= \chi_0 H_0\psi_0.\end{aligned}$$

4. Find (or guess) the function $\Lambda(\vec{r}, t)$ that corresponds to the gauge transformation in Eq. (**??**) responsible for re-expressing the vector potential in Eq. (**??**) to the form of Eq. (**??**), and show that both forms give the same magnetic field.

Solution:
Rewriting the question: Find the function $\Lambda(\vec{r}, t)$ that corresponds to the gauge transformation,

$$\vec{A}(\vec{r}, t) \rightarrow \vec{A}(\vec{r}, t) + \nabla\Lambda, \quad \Phi(\vec{r}, t) \rightarrow \Phi(\vec{r}, t) - \frac{1}{c}\frac{\partial\Lambda(\vec{r}, t)}{\partial t}$$

responsible for re-expressing the vector potential in the form

$$A_z = 0, \; A_\rho = 0, \; A_\phi = \rho B/2,$$

to the form

$$A_y = Bx, \; A_x = 0, \; A_z = 0.$$

a) For the first form,

$$\begin{aligned} A_\phi &= (1/2)B\rho, \\ A_x &= -A_\phi \sin\phi, \quad A_y = A_\phi \cos\phi, \quad A_z = 0, \\ A_x &= (-1/2)B\rho\frac{x}{\rho} = -(1/2)By, \\ A_y &= (1/2)B\rho\frac{y}{\rho} = (1/2)Bx \end{aligned}$$

Let $\Lambda = (-1/2)Bxy$,

$$\begin{aligned} \vec{A}' &= \vec{A} + (1/2)\nabla(Bxy), \\ A'_x &= 0, \quad A'_y = Bxy. \end{aligned}$$

b) For the first form

$$B_z = \partial_x A_y - \partial_y A_x = \partial_x(Bx/2) - \partial_y(-By/2) = B.$$

For the second form

$$B_z = \partial_x A_y - \partial_y A_x = \partial_x(Bx) = B. \quad \checkmark \tag{0.1}$$

5. The expression for the $\bar{v}_y$ in Eq. (**??**) is only valid for non-relativistic velocities, where $|E| << |B|$. For a uniform magnetic field $B\hat{z}$, with no electric field, consider the form for the vector potential in Eq. (**??**). Performing a relativistic boost (Lorentz transformation), but for non-relativistic velocities, in the y direction by a velocity v_y, what is the resulting zero$^{\text{th}}$ component of the vector potential A_0? Equating this with the electric scalar potential, express the strength of the resulting electric field in terms of v_y and B.

Solution:
In lab frame

$$A^0 = \Phi = -Ex, A^y = Bx.$$

Boosted

$$\begin{aligned} A^{0\prime} &= \gamma\Phi + \gamma v A^y \\ &= \gamma x(-E + v_y B). \end{aligned}$$

Choose $v_y = E/B$ to make electric field disappear. Thus, in this frame the motion is purely circular. Whereas, in the lab frame the average velocity is $v_x = E/B$, which matches our previous result.

6. In this problem, we reconsider the problem of a charged particle in the presence of both an electric and magnetic field, but do so in a different gauge. The electron is placed in a region of constant external magnetic field B directed along the z axis and of constant electric field E in the y direction.

   (a) Choosing the vector potential to lie along the y axis and describe both the electric and magnetic fields, show that the Hamiltonian may be written in the form,

$$H = \frac{P_z^2}{2m} + \frac{P_x^2}{2m} + \frac{1}{2}m\omega^2(x - x_0 - v_0 t)^2 \ ,$$

   and find ω, and v_0 in terms of E, B, e, m, k_y and c, where $\hbar k_y$ is the eigenvalue of P_y. Hint: Choose a gauge such that $\vec{E} = -(1/c)\partial_t \vec{A}$.

   (b) Show that Schrödinger's equation, $i(\partial/\partial t)\Psi = H\Psi$ is satisfied by the form

$$\Psi(x, y, z, t) = e^{-i\epsilon_n t/\hbar + imv_0 x/\hbar + ik_z z + ik_y y}\phi_n(x - x_0 - v_0 t) \ ,$$

   where ϕ_n refers to a harmonic-oscillator wave function characterized by the frequency ω and $\epsilon_n = (n + 1/2)\hbar\omega + mv_0^2/2$.

Solution:
a) Choose the gauge

$$A_y = Bx - cEt, \quad A_x = A + z = 0.$$

This will give $\vec{B} = B\hat{z}$ and $\vec{E} = E\hat{y}$. The Hamiltonian is then

$$\begin{aligned} H &= \frac{1}{2m}\left\{P_x^2 + P_z^2 + (P_y - eBx/c + eEt)^2\right\} \\ &= \frac{1}{2m}\left\{P_x^2 + P_y^2 + (p_y - eBx/c + eEt)^2\right\}, \end{aligned}$$

where we have assumed that the solution is an eigenstate of P_y with eigenvalue p_y. H is now

$$\begin{aligned} H &= \frac{1}{2m}(P_x^2 + P_z^2) + \frac{e^2B^2}{2mc^2}\left[x - (E/B)ct - p_y c/(eB)\right]^2, \\ &= \frac{1}{2m}(P_x^2 + P_z^2) + \frac{m\omega^2}{2}\left[x - x_0 - v_0 t\right]^2, \\ \omega &= \frac{eB}{m}, \quad x_0 = p_y c/(eB), \quad v_0 = (E/B)c. \end{aligned}$$

b) Applying the Hamiltonian to the form for Ψ above,

$$\begin{aligned} H\Psi = e^{-i\epsilon_n t/\hbar + imv_0 x/\hbar + ik_z z + ik_y y}&\left\{\frac{\hbar^2 k_z^2}{2m} + \frac{1}{2}mv_0^2 + H\right\}\phi_n(x - x_0 - v_0 t) \\ &- i\hbar v_0 e^{-i\epsilon_n t/\hbar + imv_0 x/\hbar + ik_z z + ik_y y}\partial_x\phi_n(x - x_0 - v_0 t) \\ &= \epsilon_n\Psi - i\hbar v_0 e^{-i\epsilon_n t/\hbar + imv_0 x/\hbar + ik_z z + ik_y y}\partial_x\phi_n(x - x_0 - v_0 t) \end{aligned}$$

Next, look at $i\hbar\partial_t\Psi$,

$$\begin{aligned} i\hbar\partial_t\Psi &= \epsilon_n\Psi + e^{-i\epsilon_n t/\hbar + imv_0x/\hbar + ik_zz + ik_yy}\partial_t\phi_n(x - x_0 - v_0t) \\ &= \epsilon_n\Psi - i\hbar e^{-i\epsilon_n t/\hbar + imv_0x/\hbar + ik_zz + ik_yy}v_0\partial_x\phi_n(x - x_0 - v_0t). \end{aligned}$$

The two expressions are identical. ✓