Chapter 2 – Homework Solutions

1. Proof that $\hbar = 0$: Consider a normalized momentum eigenstate of the momentum operator $|q\rangle$, i.e. $\mathcal{P}|q\rangle = q|q\rangle$ and $\langle q|\mathcal{P} = \langle q|q$. Consider the expectation,

$$\begin{aligned}\langle q|(\mathcal{P}\mathcal{X} - \mathcal{X}\mathcal{P})|q\rangle &= \langle q|(q\mathcal{X} - \mathcal{X}q)|q\rangle \\ &= q\langle q|(\mathcal{X} - \mathcal{X})|q\rangle = 0.\end{aligned}$$

However the commutation relation, $\mathcal{P}\mathcal{X} - \mathcal{X}\mathcal{P} = -i\hbar$, so we also have

$$\langle q|(\mathcal{P}\mathcal{X} - \mathcal{X}\mathcal{P})|q\rangle = -i\hbar.$$

Comparing the two equations, $\hbar = 0$.

What went wrong?

Solution:
This will be discussed in class.

2. Prove that the average kinetic energy is always positive, i.e.

$$\langle -\frac{\hbar^2 \partial_x^2}{2m} \rangle = -\frac{\hbar^2}{2m} \int dx\ \psi^*(x) \partial_x^2 \psi(x) > 0.$$

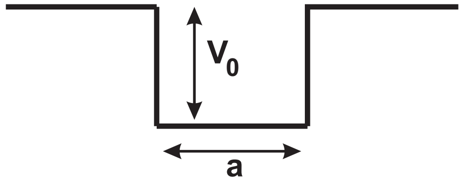
Solution:

$$\begin{aligned} \langle KE \rangle &= -\frac{\hbar^2}{2m} \int dx\ \psi^*(x) \partial_x^2 \psi(x) > 0, \\ &= \frac{\hbar^2}{2m} \int dx\ (\partial_x \psi^*(x) \partial_x \psi(x)) \\ &= \frac{\hbar^2}{2m} \int dx\ |\partial_x \psi(x)|^2 > 0. \end{aligned}$$

The first step involved integrating by parts.

3. Consider the one-dimensional potential,

$$V(x) = \begin{cases} 0, & x < -a \\ -V_0, & -a < x < a \\ 0, & x > a \end{cases}$$



For fixed a, find the minimum V_0 for the number of bound states to equal or exceed 1,2,3....

Solution:

For even parity solutions:

$$\begin{aligned} \Psi_I = \cos(k_m x), \quad \Psi_{II} = Ae^{-qx}, \\ \cos(k_m a) = Ae^{-qa}, \\ -k_m \sin(k_x a) - qAe^{-qa}, \\ k_m \tan(k_m a) = q. \end{aligned}$$

For bound state to barely exist, $q \to 0$. This gives

$$k_m a = n\pi, \quad n = 0, 1, 2, 3 \cdots .$$

For odd parity solutions,

$$\begin{aligned} \Psi_I = \sin(k_m x), \quad \Psi_{II} = Ae^{-qx}, \\ \sin(k_m a) = Ae^{-qa}, \\ k_m \cos(k_x a) - qAe^{-qa}, \\ k_m \cot(k_m a) = -q. \end{aligned}$$

The solutions disappear when

$$k_m a = (m + 1/2)\pi.$$

Thus, the N^{th} solution of any parity exists for

$$\begin{aligned} k_N a = N\pi/2, \quad N = 0, 1, 2 \cdots \\ k_N = \sqrt{2mV_0/\hbar^2}. \end{aligned}$$

The $N = 0$ solution exists for any non-zero depth. For $n > 1$ solutions,

$$\begin{aligned} a\sqrt{2mV_0/\hbar^2} = (n-1)\pi/2, \\ a \geq \frac{(n-1)\hbar\pi/2}{\sqrt{2mV_0}}. \end{aligned}$$

4. Consider a particle of mass m under the influence of the potential,

$$V(x) = V_0\theta(-x) - \frac{\hbar^2}{2m}\beta\delta(x-a), \quad V_0 \to \infty,\ \beta > 0.$$

(a) Find the transcendental equation for the energy of a bound state?

Solution:
Energy is $-\hbar^2 q^2/2m$.

$$\begin{aligned}
\psi_I(x) &= A\sinh(qx), \quad \Psi_{II}(x) = e^{-qx}, \\
A\sinh(qa) &= e^{-qa}, \\
-qe^{-qa} - qA\cosh(qa) &= -\beta e^{-qa}, \\
\frac{1}{q}\tanh(qa) &= \frac{1}{\beta - q}.
\end{aligned}$$

Solve for q.

(b) What is the minimum value of β for a ground state?

Solution:
Set $q = 0$,

$$\begin{aligned}
a &= \frac{1}{\beta}, \\
\beta &= \frac{1}{a}.
\end{aligned}$$

(c) For increasing β can one find more than one bound state?

Solution:
No, functional form does not allow more nodes. Or, you can look at graphical form of the transcendental equation.

5. Consider a plane wave moving in the $-\hat{x}$ direction to be reflected off the delta function potential, For$(x > a)$ the plane wave will have the form

$$e^{-ikx} - e^{2i\delta}e^{ikx}.$$

(a) Find the phase shift δ as a function of ka, and plot for $\beta a = 0.5$ and for $0 < ka < 10$. Because addition of $n\pi$ to the phase shift is arbitrary, translate all phases to angles between zero and π.

(b) Repeat for $\beta a = 0.99, 1.01, 1.5$.

Solution:
The wave functions in the two regions are:

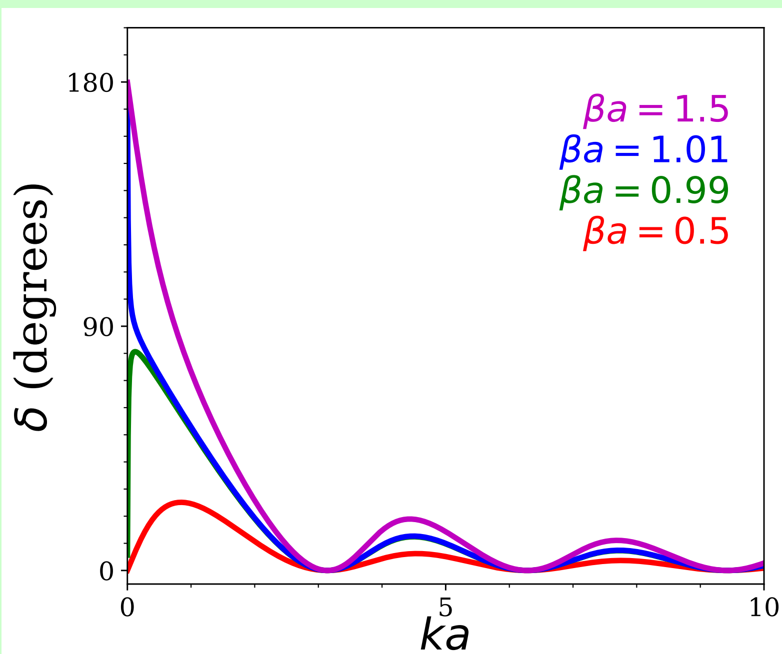
$$\psi_I = \sin(kx), \quad \psi_{II} = A\sin(kx + \delta)$$

Note that in region II we have factored out a $e^{i\delta}$ from the given form. The BC are

$$\sin(ka) = A\sin(ka + \delta),$$
$$kA\cos(ka + \delta) - k\cos(ka) = \beta\sin(ka).$$

The 2 unknowns are A and δ. Solving for δ,

$$\tan(ka + \delta) = k\frac{\sin(ka)}{\beta\sin(ka) + k\cos(ka)},$$
$$\delta = -ka + \tan^{-1}\{frac\sin(ka)(\beta/k)\sin(ka) + \cos(ka)\}$$



6. Consider a particle of mass m interacting with a repulsive δ function potential,

$$V(x) = \frac{\hbar^2}{2m}\beta\delta(x).$$

Consider particles of energy E incident on the potential.

(a) What fraction of particles are reflected by the potential?

(b) Show that the currents for$x >$ and for $x < 0$ are the same.

Solution:
a)

$$\psi_I = e^{ikx} + Ae^{-ikx}, \quad \psi_{II} = Be^{ikx},$$

B.C.:

$$\begin{aligned} 1 + A &= B, \\ ikB - ik + ikA &= -\beta B. \end{aligned}$$

Solving for A,

$$\begin{aligned} 1 + A &= \frac{ikA}{-\beta - ik}, \\ -ik - \beta A &= ikA, \\ A &= \frac{-ik}{ik + \beta}, \\ |A|^2 &= \frac{k^2}{k^2 + \beta^2}. \end{aligned}$$

b) Solving for B,

$$\begin{aligned} B = 1 + A &= \frac{\beta}{ik + \beta}, \\ |B|^2 &= \frac{\beta^2}{k^2 + \beta^2}. \end{aligned}$$

The currents are

$$\begin{aligned} j(x > 0) &= k\frac{\beta^2}{k^2 + \beta^2}, \\ j(x < 0) &= \mathrm{Re}\left\{\left(e^{-ikx} + A^* e^{ikx}\right)\left(ke^{ikx} - kAe^{-ikx}\right)\right\}, \\ &= \mathrm{Re}\left\{k - k|A|^2 + kA^* e^{ikx} - kAe^{-ikx}\right\} \\ &= k(1 - |A|^2) = k|B|^2 = k\frac{\beta^2}{k^2 + \beta^2} \quad \checkmark \end{aligned}$$

7. Consider a three-dimensional harmonic oscillator with quantum numbers n_x, n_y and n_z. How many states are there with a given $N = n_x + n_y + n_z$? Find a closed expression (no sum). Test it for all $n \leq 3$.

Solution:
First, for $N_{\text{states},xy}$, the number of states where n_x+n_y adds to $N_{rmstates,xy}$ is (defining $n \equiv n_x+n_y$)

$$N_{\text{states},xy} = n_x + n_y + 1 = n + 1.$$

The number of ways to add to $N = n + n_z$ is

$$\begin{aligned} N_{\text{states}} &= \sum_{n=0}^{N} N_{\text{states},xy} \\ &= \sum_{n=0}^{N} (n+1) = \frac{N(N+1)}{2} + N + 1 \\ &= \frac{(N+1)(N+2)}{2}. \end{aligned}$$

8. Calculate $\langle 0|aaa^\dagger aa^\dagger a^\dagger|0\rangle$ and $\langle n|a^\dagger a^\dagger a^\dagger a|m\rangle$.

Solution:

$$
\begin{aligned}
\langle 0|aaa^\dagger aa^\dagger a^\dagger|0\rangle &= \langle 0|(aa)N(a^\dagger a^\dagger)|0\rangle \\
&= 2\langle 0|(aa)N(a^\dagger a^\dagger)|0\rangle \\
&= 4, \\
\langle n|a^\dagger a^\dagger a^\dagger a|m\rangle &= \sqrt{n(n-1)(n-2)}\langle n-3|m-1\rangle\sqrt{m} \\
&= \sqrt{n(n-1)(n-2)m}\,\delta_{n-3,m-1} \\
&= \delta_{n-2,m}(n-2)\sqrt{n(n-1)}.
\end{aligned}
$$

9. Find $\psi_1(x)$, the wave function of the first excited state by applying $a^\dagger$, defined in Eq. (**??**), to the ground state.

Solution:

$$
\begin{aligned}
|\psi_1\rangle &= a^\dagger|0\rangle, \\
a^\dagger &= \sqrt{\frac{m\omega 2\hbar}{X}} - i\sqrt{\frac{1}{2\hbar m\omega}}P, \\
\psi_0(x) &= Z^{-1/2}e^{-x^2/2b^2}, \quad Z = \pi^{1/2}b, b = \sqrt{\frac{\hbar}{m\omega}}, \\
\psi_1(x) &= \frac{1}{\sqrt{Z}}\left\{\sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}(-i\hbar)\partial_x\right\}e^{-x^2/2b^2} \\
&= Z^{-1/2}\left\{\sqrt{\frac{m\omega}{2\hbar}}X + \sqrt{\frac{\hbar}{2m\omega}}\frac{x}{b^2}\right\}e^{-x^2/2b^2} \\
&= \frac{x}{\sqrt{Z}}\sqrt{2m\omega}\hbar e^{-x^2}2b^2 \\
&= \sqrt{\frac{2}{\pi^{1/2}}}\frac{x}{b^{3/2}}e^{-x^2/2b^2}.
\end{aligned}
$$

10. Consider a particle of mass m in a harmonic oscillator with spring constant $k = m\omega^2$.

   (a) Write the momentum and position operators for a particle of mass m in a harmonic oscillator characterized by frequency ω in terms of the creation and destruction operators.
   (b) Calculate $\langle n|\mathcal{X}^2|n\rangle$ and $\langle n|\mathcal{P}^2|n\rangle$. Compare the product of these two matrix elements to the constraint of the uncertainty relation as a function of n.
   (c) Show that the expectation value of the potential energy in an energy eigenstate of the harmonic oscillator equals the expectation value of the kinetic energy in that state.

Solution:
a)

$$\begin{aligned}
a^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P, \\
a &= \sqrt{\frac{m\omega}{2\hbar}}X + i\sqrt{\frac{1}{2\hbar m\omega}}P, \\
X &= \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \\
P &= i\sqrt{\frac{\hbar m\omega}{2}}(a^d agger - a).
\end{aligned}$$

b)

$$\begin{aligned}
\langle n|X^2|n\rangle &= \frac{\hbar}{2m\omega}\langle n|(a + a^\dagger)^2|n\rangle \\
&= \frac{\hbar}{2m\omega}\langle n|aa^\dagger + a^\dagger a|n\rangle \\
&= \frac{\hbar}{2m\omega}(2n+1), \\
\langle n|P^2|n\rangle &= \frac{\hbar m\omega}{2}(2n+1) \\
\langle n|X^2|n\rangle\langle|P^2|n\rangle &= (2n+1)^2\frac{\hbar^2}{4}.
\end{aligned}$$

For ground state $= \hbar^2/4$ as expected. c)

$$\begin{aligned}
\langle n|\frac{P^2}{2m}|n\rangle &= \frac{\hbar\omega}{4}(2n+1), \\
\langle n|\frac{1}{2}m\omega^2 X^2|n\rangle &= \frac{\hbar\omega}{4}(2n+1) \quad \checkmark
\end{aligned}$$

11. (a) What is the representation of the position operator in the momentum basis – how is $\langle p|\mathcal{X}|\Psi\rangle$ related to $\langle p|\Psi\rangle$? Use the completeness relation, $\int dx|x\rangle\langle x| = \mathbb{I}$ and the fact that $\langle p|x\rangle = e^{-ipx/\hbar}$.

(b) Suppose that the potential is $v(\mathbf{x}) = (k/2)x^2$. What is the Schrödinger equation written in momentum space; i.e. what is the equation of motion of the amplitude $\langle p|\Psi(t)\rangle$?

Solution:

a)

$$\begin{aligned}\langle p|X|\psi\rangle &= \int dx\ \langle p|x\rangle x\langle x|\psi\rangle \\ &= i\hbar\partial_p \int dx\ \langle p|x\rangle\langle x|\psi\rangle, \\ &= i\hbar\partial_p\langle p|\psi\rangle.\end{aligned}$$

b)

$$\begin{aligned}H &= -\frac{k\hbar^2}{2}\partial_p^2 + \frac{p^2}{2m}, \\ H\psi(p) &= E\psi(p).\end{aligned}$$

It looks just like a harmonic oscillator form.

12. Consider a potential

$$V(x) = \begin{array}{cc} 0, & x < -a \\ u(x), & -a < x < a \\ 0, & x > a \end{array}$$

where $u(x)$ is an arbitrary real function. Consider a wave incident from the left. Suppose that the transmission amplitude, defined as the ratio of the transmitted wave at $x = a$ to the incident wave at $x = -a$, is $S(E)$. Now consider a wave incident from the right. Show that the transmission amplitudes, $|S(E)|$, are the same for both directions. *(Hint: the Schrödinger equation in this case is a real equation, so the complex conjugate of a solution is also a solution.)*

Solution:
The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\partial_x^2\psi(x) + u(x)\psi(x) = E\psi(x),$$
$$\psi(x < -a) = e^{ikx} + Be^{-ikx},$$
$$\psi(x > a) = Ce^{ikx}.$$

The transmission amplitude is C. Because the Hamiltonian is real, you can take the complex conjugate of this solution and get another solution with the same energy,

$$\phi(x < -a) = e^{-ikx} + B^* e^{ikx},$$
$$\phi(x > a) = C^* e^{-ikx}.$$

Now consider a linear combination of the two solutions, $\chi = B^*\psi - \phi$,

$$\chi(x < -a) = (B^*B - 1)e^{-ikx},$$
$$\chi(x > a) = B^*Ce^{ikx} - C^*e^{-ikx}$$

The transmission amplitude for going right to left is

$$S(E) = \frac{B^*B - 1)}{C^*} = -\frac{|C|^2}{-C^*} = C,$$

where the fact that $|B|^2 + |C|^2 = 1$ was used. The squared amplitudes are then equal.

13. (a) Derive and solve the equations of motion for the Heisenberg operators $a(t)$ and $a^\dagger(t)$ for the harmonic oscillator.

(b) Calculate $[a(t), a^\dagger(t')]$.

Solution:

a)

$$\begin{aligned}
\frac{d}{dt}a(t) &= \frac{d}{dt}\left\{e^{iHt/\hbar}ae^{-iHt/\hbar}\right\} \\
&= \frac{i}{\hbar}e^{iHt/\hbar}[H,a]e^{-iHt/\hbar} \\
H &= \hbar\omega(a^\dagger a + 1/2), \\
[H,a] &= \hbar\omega(a^\dagger aa - aa^\dagger a) \\
&= \hbar\omega(a^\dagger aa - a^\dagger aa - a) \\
&= -\hbar\omega a, \frac{d}{dt}a(t) && = -i\omega a(t).
\end{aligned}$$

Similarly,

$$\frac{d}{dt}a^\dagger(t) = i\omega a^d agger(t).$$

Solutions to the equations of motion are:

$$\begin{aligned}
a(t) &= e^{-i\omega t}a, \\
a^\dagger(t) &= e^{i\omega t}
\end{aligned}$$

b)

$$[a(t), a^\dagger(t')] = e^{i\omega(t-t')}.$$

14. Calculate the correlation function $\langle 0|x(t)x(t')|0\rangle$ for the harmonic oscillator where $|0\rangle$ is the harmonic oscillator ground state, and $x(t)$ is the position operator in the Heisenberg representation. Hint: use the expressions for $a(t)$ and $a^\dagger(t)$ from the previous problem. Then solve for the equations of motion for both $x(t)$ and $p(t)$.

Solution:
From previous problem,

$$\begin{aligned} a(tY) &= e^{-i\omega t}a, \quad a^\dagger(t) = e^{i\omega t}a^\dagger, \\ x(t) &= \sqrt{\frac{\hbar}{2m\omega}}\left[e^{-i\omega t}a + e^{i\omega t}a^\dagger\right], \\ \langle 0|x(t)x(t')|0\rangle &= \frac{\hbar}{2m\omega}\langle 0|(e^{-i\omega t}a + e^{i\omega t}a^\dagger)(e^{-i\omega t'}a + e^{i\omega t'}a^\dagger)|0\rangle, \\ &= \frac{\hbar}{2m\omega}e^{i\omega(t'-t)}. \end{aligned}$$

15. What are the matrix elements of the operator $1/|\vec{p}|$ in the position representation? That is, find

$$\langle \mathbf{r}|\frac{1}{|\mathbf{p}|}|\mathbf{r}'\rangle.$$

Work the problem in three dimensions.

Solution:

$$\begin{aligned}
\langle \vec{r}|\frac{1}{|\vec{p}|}|\vec{r}'\rangle &= \int \frac{d^3q d^3q'}{(2\pi)^6}\langle \vec{r}|\vec{q}\rangle\langle \vec{q}|\frac{1}{|\vec{P}|}|\vec{q}'\rangle\langle \vec{q}'|\vec{r}'\rangle \\
&= \int \frac{d^3q d^3q'}{(2\pi)^3} e^{i\vec{q}'\cdot\vec{r}' - i\vec{q}\cdot\vec{r}}\frac{1}{\hbar q}\delta(\vec{q}-\vec{q}') \\
&= \int \frac{d^3q}{(2\pi)^3}\frac{e^{i\vec{q}\cdot(\vec{r}-\vec{r}')}}{\hbar|\vec{q}|} \\
&= \frac{1}{4\pi^2\hbar}\int \frac{q^2 dq d\cos\theta}{q} e^{iq|\vec{r}-\vec{r}'|\cos\theta} \\
&= \frac{1}{2\pi^2\hbar}\int dq\ \frac{\sin(q|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|} \\
&= \frac{1}{2\pi^2\hbar}\frac{-1}{|\vec{r}-\vec{r}'|^2}.
\end{aligned}$$

16. Calculate the Wigner transform $f(p,x)$ for a particle in the ground state of an infinite square well potential,

$$V(x) = \begin{cases} \infty, & x < 0 \\ 0, & -a/2 < x < a/2 \\ \infty, & x > a \end{cases} .$$

Are there any regions with phase space densities either greater than unity or less than zero?

Solution:

$$\psi(x) = \sqrt{\frac{2}{a}}\cos(\pi x/a) = \cos(qx), \quad -a/2 < x < a/2,$$

Let $x > 0$,

$$\begin{aligned} f(k,x) &= \frac{2}{a}\int_{-y_{\rm max}}^{y_{\rm max}} dy\ \cos[q(x+y/2)]\cos[q(x-y/2)]e^{iky} \\ &= \frac{1}{a}\int_{-y_{\rm max}}^{y_{\rm max}} dy\ [\cos(2qx) + cos(qy]\cos(ky) \\ &= \frac{2}{ka}\cos(2qx)\sin(ky_{\rm max}) + \frac{\sin[(q+k)y_{\rm max}]}{(q+k)a} + \frac{\sin[(q-k)y_{\rm max}]}{(q-k)a}, \\ y_{\rm max} &= a - 2x,\ x > 0, \\ &= a + 2x, \quad x < 0. \end{aligned}$$