Chapter 14 – Homework Solutions

1. Consider bosonic creation and destruction operators, $a^\dagger$ and a. Consider a linear combination,

$$b = \alpha a + \beta a^\dagger$$

What is the constraint on the complex numbers α and β if one is to demand that $[b, b^\dagger] = 1$?

Solution:

$$\begin{aligned}
[b, b^\dagger] &= [(\alpha a + \beta a^\dagger), (\alpha^* a^\dagger + \beta^* a)] \\
&= |\alpha|^2 [a, a^\dagger] + |\beta^2| [a^\dagger, a] + \beta\alpha^* [a^\dagger, a^\dagger] + \alpha\beta^* [a, a] \\
&= (|\alpha|^2 - |\beta|^2)[a, a^\dagger] \\
&= |\alpha|^2 - |\beta^2| \\
&= 1.
\end{aligned}$$

This can be equivalently states as $\alpha = e^{i\gamma} \cosh \eta, \beta = e^{i\delta} \sinh \eta$, where γ, δ and η can be any real numbers.

2. Consider two oscillator levels described by the creation operators, $a_1^\dagger$ and $a_2^\dagger$, where the Hamiltonian is

$$H = \epsilon_1 a_1^\dagger a_1 + \epsilon_2 a_2^\dagger a_2 + \beta(a_1^\dagger a_2^\dagger + a_1 a_2).$$

Consider the operators

$$b_1^\dagger \equiv \cosh\eta\ a_1^\dagger + \sinh\eta\ a_2,$$
$$b_2^\dagger \equiv \cosh\eta\ a_2^\dagger + \sinh\eta\ a_1.$$

(a) Show that b_i and $b_i^\dagger$ behave like creation/destruction operators.

(b) Find the values of η, E_0, E_1 and E_2 that allow H to be written as

$$H = E_0 + E_1 b_1^\dagger b_1 + E_2 b_2^\dagger b_2.$$

This is known as a Bogoliubov transformation.

Solution:

a)

$$\begin{aligned}[b_1, b_1^\dagger] &= [\cosh\eta a_1 + \sinh\eta a_2^\dagger, \cosh\eta a_1^\dagger + \sinh\eta a_2] \\ &= \cosh^2\eta - \sinh^2\eta = 1 \quad \checkmark\end{aligned}$$

$$\begin{aligned}[b_1, b_2^\dagger] &= [\cosh\eta a_1 + \sinh\eta a_2^\dagger, \cosh\eta a_2^\dagger + \sinh\eta a_1] \\ &= \cosh\eta\sinh\eta(1-1) = 0 \quad \checkmark\end{aligned}$$

Take h.c. of previous to show $[b_2, b_1^\dagger] = 0$.

$$\begin{aligned}[b_2, b_2^\dagger] &= [\cosh\eta a_2 + \sinh\eta a_1^\dagger, \cosh\eta a_2 + \sinh\eta a_1] \\ &= \cosh^2\eta - \sinh^2\eta = 1 \quad \checkmark\end{aligned}$$

$$\begin{aligned}[b_1^\dagger, b_2^\dagger] &= [\cosh\eta a_1^\dagger + \sinh\eta a_2, \cosh\eta a_2^\dagger + \sinh\eta a_1] \\ &= -\cosh\eta\sinh\eta + \cosh\eta\sinh\eta = 0 \quad \checkmark\end{aligned}$$

Take h.c. of previous to show $[b_1, b_2] = 0$.

b) First write H in terms of $a_1, a_2, a_1^\dagger, a_2^\dagger$,

$$\begin{aligned}H &= E_0 + E_1 b_1^\dagger b_1 + E_2 b_2^\dagger b_2 \\ &= E_0 + E_1(\cosh\eta a_1^\dagger + \sinh\eta a_2)(\cosh\eta a_1 + \sinh\eta a_2^\dagger) \\ &+ E_2(\sinh\eta a_1 + \cosh\eta a_2^\dagger)(\sinh\eta a_1^\dagger + \cosh\eta a_2) \\ &= E_0 + (E_1\cosh^2\eta + E_2\sinh^2\eta)a_1^\dagger a_1 + (E_1\sinh^2\eta + E_2\cosh^2\eta)a_2^\dagger a_1^\dagger \\ &= (E_1\cosh\eta\sinh\eta + E_2\cosh\eta\sinh\eta)(a_2^\dagger a_1^\dagger + a_1 a_2) + (E_1 + E_2)\sinh^2\eta\end{aligned}$$

Equate it to the form

$$H = \epsilon_1 a_1^\dagger a_1 + \epsilon_2 a_2^\dagger a_2 + \beta(a_1^\dagger a_2^\dagger + a_1 a_2).$$

This requires

$$\begin{aligned}
(1)\ \frac{1}{2}(E_1 + E_2)\sinh 2\eta &= \beta, \\
(2)\ E_1 \cosh^2\eta + E_2 \sinh^2\eta &= \epsilon_1, \\
(3)\ E_1 \sinh^2\eta + E_2 \cosh^2\eta = \epsilon_2, (4)\ -(E_1 + E_2)\sinh^2\eta &= E_0.
\end{aligned}$$

Solving for η, E_1, E_2, E_0,

$$\begin{aligned}
(2) - (3) \rightarrow\ & E_1 - E_2 = \epsilon_1 - \epsilon_2, \\
(1) + (2) \rightarrow\ & (E_1 + E_2)\cosh 2\eta = \epsilon_1 + \epsilon_2.
\end{aligned}$$

Combine with (1)

$$\begin{aligned}
\frac{1}{2}\tanh 2\eta &= \frac{\beta}{\epsilon_1 + \epsilon_2}, \\
\eta &= \frac{1}{2}\tanh^{-1}\left(\frac{2\beta}{\epsilon_1 + \epsilon_2}\right), \\
E_1 + E_2 &= (\epsilon_1 + \epsilon_2)\mathrm{sech}2\eta, \\
E_1 &= \frac{1}{2}\left\{\epsilon_1(\mathrm{sech}2\eta + 1) + \epsilon_2(\mathrm{sech}2\eta - 1)\right\}, \\
E_2 &= \frac{1}{2}\left\{\epsilon_1(\mathrm{sech}2\eta - 1) + \epsilon_2(\mathrm{sech}2\eta - 1)\right\}, \\
E_0 &= -(E_1 + E_2)\sinh^2\eta = -(\epsilon_1 + \epsilon_2)\frac{\sinh^2\eta}{\cosh 2\eta} \\
\sinh^2\eta &= \frac{1}{2}(\cosh 2\eta - 1), \\
E_0 &= -\frac{\cosh 2\eta - 1}{2\cos 2\eta}(\epsilon_1 + \epsilon_2) \\
&= -(\epsilon_1 + \epsilon_2)(1 - \mathrm{sech}2\eta).
\end{aligned}$$

3. Consider the coherent state $|\eta\rangle$ defined by,

$$|\eta\rangle = e^{-\eta^*\eta/2} \exp(\eta a^\dagger)|0\rangle.$$

(a) Show that $|\eta\rangle$ can also be written as

$$|\eta\rangle = e^{-\eta^* a + \eta a^\dagger}|0\rangle.$$

Hint: You may wish to use the Baker-Campbell-Hausdorff lemma.

(b) Show that the overlap of two states is given by,

$$\langle\eta'|\eta\rangle = e^{-|\eta'|^2/2 - |\eta|^2/2 + \eta'^*\eta}.$$

Solution:

a) From BCH lemma, (using the fact that $[a, a^\dagger] =$ number)

$$\begin{aligned} e^{-\eta^* a + \eta a^\dagger}|0\rangle &= e^{\eta a^\dagger} e^{\eta^* a} e^{-\eta^*\eta[a,a^\dagger]/2}|0\rangle, \\ &= e^{-\eta^*\eta/2} e^{\eta a^\dagger} e^{\eta^* a}|0\rangle \\ &= e^{-\eta^*\eta/2} e^{\eta a^\dagger}|0\rangle \end{aligned}$$

b)

$$\begin{aligned} \langle\eta'|\eta\rangle &= e^{-\eta'^*\eta'/2}\langle 0|e^{\eta'^* a}|\eta\rangle \\ &= e^{-\eta'^*\eta'/2} e^{\eta'^*\eta}\langle 0|\eta\rangle \\ &= e^{-\eta'^*\eta'/2} e^{\eta'^*\eta} e^{-\eta^*\eta/2}. \end{aligned}$$

4. Consider a coherent state

$$|\eta\rangle = e^{-\eta^*\eta/2} e^{\eta a^\dagger} |0\rangle.$$

   (a) Show that $\bar{N} = \langle\eta|N_{\text{op}}|\eta\rangle = \eta^*\eta$, where $N_{\text{op}} = a^\dagger a$ is the number operator.

   (b) Show that the variance equals the mean, i.e.,

$$\langle\eta|(N_{\text{op}} - \bar{N})^2|\eta\rangle = \bar{N}.$$

   This is characteristic of a Poissonian distribution.

Solution:
a) Because $|\eta\rangle$ is an eigenstate of a,

$$\langle\eta|a^\dagger a|\eta\rangle = \eta^*\eta\langle\eta|\eta\rangle = \eta^*\eta.$$

b)

$$\begin{aligned}
&\langle\eta|((a^\dagger a - \bar{N})^2|\eta\rangle \\
&\qquad = \langle\eta|(a^\dagger a a^\dagger a|\eta\rangle + \bar{N}^2 - 2\bar{N}\langle\eta|a^\dagger a|\eta\rangle \\
&\qquad = \langle\eta|(a^\dagger a a^\dagger a|\eta\rangle - \bar{N}^2 \\
&\qquad = \langle\eta|(a^\dagger a^\dagger a a|\eta\rangle + \langle\eta|a^\dagger a|\eta\rangle - \bar{N}^2 \\
&\qquad = (\eta^*)^2\eta^2 - \bar{N}^2 + \bar{N} \\
&\qquad = \bar{N}.
\end{aligned}$$