Chapter 13 – Homework Solutions

1. To show why derivatives are defined as shown in Eq. (**??**), show that

$$\partial_\mu x^2 = 2x_\mu, \quad \text{and } \partial^\mu x^2 = 2x^\mu,$$

where $x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$.

Solution:

$$\begin{aligned}
x^2 &= (x^0)^2 - \sum_i (x^i)^2, \\
\frac{\partial}{\partial x^\mu} &= \left\{ \begin{array}{rl} 2x^0, & \mu = 0 \\ -2x^i, & \mu = i \end{array} \right. \\
&= 2x_\mu.
\end{aligned}$$

Thus $\partial/\partial x^\mu$ behaves as a covariant (subscript) vector and should be represented as ∂_μ. Now, one can repeat with the change that one takes the derivative w.r.t. x_μ,

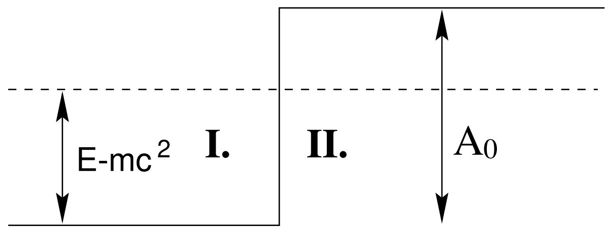
$$\begin{aligned}
x^2 &= (x^0)^2 - \sum_i (x^i)^2, \\
\frac{\partial}{\partial x_i} &= -\frac{\partial}{\partial x^i}, \\
\frac{\partial}{\partial x_\mu} &= \left\{ \begin{array}{rl} 2x^0, & \mu = 0 \\ 2x^i, & \mu = i \end{array} \right. \\
&= 2x^\mu.
\end{aligned}$$

This shows that $\partial/\partial x_\mu$ behaves as a contravariant vector and should be represented as ∂^μ.

2. Consider a charged relativistic particle interacting with the electromagnetic field, and described by the Klein-Gordon equation.

$$\left[(i\hbar\partial_t - eA_0)^2 + c^2\hbar^2\partial_x^2 - m^2c^4\right]\psi(x,t) = 0$$

The electrostatic potential A_0 is illustrated in the diagram below.



Consider a solution for a particle incident from the left,

$$\begin{aligned}\psi_I(x,t) &= e^{(-iEt+ikx)/\hbar} + Be^{(-iEt-ikx)/\hbar}\\ \psi_{II}(x,t) &= Ce^{(-iEt+ik'x)/\hbar},\end{aligned}$$

where $E = \sqrt{m^2c^4 + k^2}$.

Calculate the charge and current densities (include direction) in regions I and II for each of the following three cases.

(a) $eA_0 < E - mc^2$.

(b) $E - mc^2 < eA_0 < E + mc^2$.

(c) $eA_0 > E + mc^2$.

Solution:
a) The momenta on the right is

$$\begin{aligned}k' &= \sqrt{(E - eA_0)^2 - m^2},\ k' \text{ is real}\\ E - eA_0 &> mc^2.\end{aligned}$$

First BC

$$k(1 - B) = k'C.$$

Second BC

$$\begin{aligned}C(1 + k'/k) &= 2,\\ C &= 2k/(k + k')\\ B &= (k = -k')/(k + k')\end{aligned}$$

Calculate currents and densities in region I

$$\begin{aligned}
\rho_I &= E\left|e^{ikx} + \frac{(k-k')}{k+k'}e^{-ikx}\right|^2, \\
&= E\left\{1 + \frac{(k-k')^2}{(k+k')^2} + 2\frac{(k-k')}{(k+k')}\cos(2kx)\right\}, \\
j_I &= \frac{1}{2}\left(e^{-ikx} + \frac{k-k'}{k+k'}e^{ikx}\right)(-i\partial_x)\left(e^{ikx} + \frac{k-k'}{k+k'}e^{-ikx}\right) \\
&+ \frac{1}{2}\left[-i\partial_x\left(e^{-ikx} + \frac{k-k'}{k+k'}e^{ikx}\right)\right]\left(e^{ikx} + \frac{k-k'}{k+k'}e^{-ikx}\right) \\
&= \frac{k}{2}\left\{1 - \frac{(k-k')^2}{(k+k')^2} + 2i\frac{(k-k')}{(k+k')}\sin(2kx)\right\} + \frac{k}{2}\left\{1 - \frac{(k-k')^2}{(k+k')^2} - 2i\frac{(k-k')}{(k+k')}\sin(2kx)\right\} \\
&= k\left\{1 - \frac{(k-k')^2}{(k+k')^2}\right\} \\
&= \frac{4k^2k'^2}{(k+k')^2}.
\end{aligned}$$

Now, do the same in region II

$$\begin{aligned}
\rho_{II} &= (E - eA_0)\frac{4k^2}{(k+k')^2}, \\
j_{II} &= k'\frac{4k^2}{(k+k')^2}.
\end{aligned}$$

Indeed, the currents are the same on both sides of the barrier.

b)

$$\begin{aligned} q' &= \sqrt{m^2 - (E - eA_0)^2} \ \text{ is positive and real} \\ \Psi_{II} &= Ce^{-q'x}. \end{aligned}$$

First BC

$$1 + B = C$$

Second BC

$$k(1 - B) = iq'C.$$

Solving the BC

$$\begin{aligned} C &= \frac{2k}{k + iq'}, \\ B &= \frac{k - iq'}{k + iq'}, \quad \text{note } B^*B = 1. \end{aligned}$$

The currents and densities are

$$\begin{aligned} \rho_I &= E\left|e^{ikx} + Be^{-ikx}\right|^2 = E\left\{2 + Be^{-2ikx} + B^*e^{2ikx}\right\} \\ &= E\left\{2 + 2B_R\cos(2kx) + 2B_I\sin(2kx)\right\}, \\ j_I &= \frac{1}{2}\left(e^{-ikx} + B^*e^{ikx}\right)(-i\partial_x)\left(e^{ikx} + Be^{-ikx}\right) + h.c. \\ &= \left(k - k + kB^2e^{2ikx} - kBe^{-2ikx}\right) + h.c. \\ &= 0, \\ \rho_{II} &= \frac{4k^2}{k^2 + q'^2}(E - eA_0)e^{-2q'x}, \\ j_{II} &= \frac{1}{2}\frac{4k^2}{k^2 + q'^2}\left\{ike^{-2q'x} + h.c\right\} \\ &= 0. \end{aligned}$$

c)

$$k' = \sqrt{(E - eA_0)^2 - m^2} \quad \text{is real.}$$

Everything is same as in (a)

$$\begin{aligned} \rho_I &= E\left\{1 + \frac{(k - k')^2}{(k + k')^2} + 2\frac{(k - k')}{(k + k')}\cos(2kx)\right\}, \\ j_I &= \frac{4k^2k'^2}{(k + k')^2}, \\ \rho_{II} &= (E - eA_0)\frac{4k^2}{(k + k')^2}, \\ j_{II} &= k'\frac{4k^2}{(k + k')^2}. \end{aligned}$$

Note that $\rho_{II}$is negative, but $j_I = j_{II}$ is positive. This represents anti-particles approaching from right. The annihilate with particles coming in from left to produce particles reflected back to left.

3. Consider the same case as above, except with no electrostatic potential. Instead, consider a different mass in region I and region II, with $m_{II} > m_I$. For each of the following two cases, calculate the charge and current densities in regions I and II.

   (a) $E > m_{II}c^2$

   (b) $E < m_{II}c^2$

Solution:
Region I

$$
\begin{aligned}
E^2 k^2 + m^2, \quad m = m_I, \\
\psi_I = e^{ikx} + Be^{-ikx}.
\end{aligned}
$$

Region II

$$
\begin{aligned}
E^2 &= k'^2 + (m+\Phi)^2, \quad \Phi = m_{II} - m_I, \\
k' &= \sqrt{E^2 - (m+\Phi)^2}, \\
\psi_{II} &= Ce^{ik'x}.
\end{aligned}
$$

a) k' is real.

$$
\begin{aligned}
1 + B &= C, \\
k(1-B) &= k'C, \\
C &= \frac{2}{1+k'/k} \\
&= \frac{2k}{(k+k')}, \\
B &= \frac{k-k'}{k+k'}.
\end{aligned}
$$

Calculating densities and currents

$$
\begin{aligned}
\rho_I &= E\left\{1 + \frac{(k-k')^2}{(k+k')^2} + 2\frac{(k-k')}{(k+k')}\cos(2kx)\right\}, \\
j_I &= k(e^{-ikx} + B^* e^{ikx})(e^{ikx} - Be^{-ikx}) + h.c. \\
&= k\{1 - |B|^2\} \\
&= \frac{4k^2k'}{(k+k')^2}, \\
\rho_{II} &= E|C|^2 \\
&= E\frac{4k^2}{(k+k')^2}, \\
j_{II} &= k'|C|^2 \\
&= \frac{4k'k^2}{(k+k')^2}
\end{aligned}
$$

b) k' is imaginary

$$
\begin{aligned}
q' &= \sqrt{(m+\Phi)^2 - E^2}, \\
\psi_{II} &= C e^{-q'x}, \\
C &= \frac{2k}{k+iq'}, \\
B &= \frac{k-iq'}{k+iq'}, \quad |B|^2 = 1, \\
\rho_1 &= E\left\{(e^{-ikx} + B^* e^{ikx})(e^{ikx} + B e^{-ikx})\right\} \\
&= E\left\{2 + B e^{-2ikx} + B^* e^{2ikx}\right\} \\
j_I &= \frac{k}{2}\left\{(e^{-ikx} + B^* e^{ikx})(e^{ikx} - B e^{-ikx})\right\} + h.c. \\
&= \frac{k}{2}\left\{1 - 1 - B e^{-2ikx} + B^* e^{2ikx}\right\} + h.c. \\
&= 0, \\
\rho_{II} &= |C|^2 E e^{-2q'x}, \\
&= \frac{4k^2}{k^2 + q'^2} e^{-2q'x}, \\
j_{II} &= 0.
\end{aligned}
$$

4. Consider the Dirac representation,

$$\beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \qquad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

and the chiral representation,

$$\beta = \begin{pmatrix} 0 & -\mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \qquad \vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}$$

The spinors, $u_\uparrow$ and $u_\downarrow$, represent positive-energy eigenvalues of the Dirac equation assuming the momentum is along the z axis.

$$(m\beta + p_z\alpha_z)\, u(p_z) = Eu(p_z)\ ,$$

The spin labels, $\uparrow$ and $\downarrow$ refer to the positive and negative values of the spin operator, which in both representations is

$$\Sigma_z = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$$

Write the four-component spinors $u_\uparrow$ and $u_\downarrow$ in terms of p, E and m :

(a) in the Dirac representation.
(b) in the chiral representation.
(c) in the limit $p_z \to 0$ for both representations.
(d) in the limit $p_z \to \infty$ for both representations.

Solution:
To be have a +1 eigenvalue of Σ_z, the states must be of the form

$$u_\uparrow = \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} = e^{-iEt+ipz}.$$

To have a -1 eigenvalue of Σ_z,

$$u_\downarrow = \begin{pmatrix} 0 \\ a \\ 0 \\ b \end{pmatrix} e^{-iEt+ipz}.$$

We need to find states such that

$$(m\beta + p\alpha_z)u = Eu$$

a) Dirac rep.

$$\beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \quad \alpha_z \qquad = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}.$$

Setting $Hu_\uparrow = Eu_\uparrow$,

$$
\begin{aligned}
ma + pb &= Ea, \\
-mb + pa &= Eb, \\
a &= b\frac{p}{E-m}, \\
u_\uparrow &= \frac{p}{\sqrt{2E^2 - 2mE}} \begin{pmatrix} 1 \\ 0 \\ p/(E+m) \\ 0 \end{pmatrix}, \quad \text{normalized}
\end{aligned}
$$

Now for $u_\downarrow$,

$$
\begin{aligned}
ma - pb &= Ea, \\
-mb - pa &= Eb, \\
a &= b\frac{-p}{E-m}, \\
u_\downarrow &= \frac{p}{2E^2 - 2mE} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -p/(E+m) \end{pmatrix}.
\end{aligned}
$$

b) In the chiral representation. For $u_\uparrow$,

$$
\begin{aligned}
-mb + pa &= Ea, \\
-ma - pb &= Eb, \\
a &= \frac{m}{E-p}b, \\
u_\uparrow &= \frac{m}{\sqrt{2E^2 - 2Ep}} \begin{pmatrix} 1 \\ 0 \\ -(E-p)/m \\ 0 \end{pmatrix}.
\end{aligned}
$$

For $u_\downarrow$,

$$
\begin{aligned}
-mb - pa &= Ea, \\
-ma + pb &= Eb, \\
a &= -\frac{(E-p)}{m}b, \\
u_\downarrow &= \frac{m}{\sqrt{2E^2 - 2Ep}} \begin{pmatrix} 0 \\ (E-p)/m \\ 0 \\ 1 \end{pmatrix}.
\end{aligned}
$$

c) As $p \to 0$,
Dirac representation,

$$u_\uparrow = \frac{p}{\sqrt{2m(E-m)}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{p\sqrt{E+m}}{\sqrt{2m(E-m)(E+m)}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Similarly,

$$u_\downarrow = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

In the chiral representation

$$u_\uparrow = \frac{m}{\sqrt{2m^2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

Similarly

$$u_\downarrow = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

d) As $m \to 0$, In Dirac representation

$$u_\uparrow = \frac{p}{\sqrt{2p^2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

Similarly,

$$u_\downarrow = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix},$$

In Chiral representation,

$$u_\uparrow = \frac{m}{\sqrt{2E(E-p)}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$= \frac{m\sqrt{2p}}{\sqrt{2E(E-p)(E+p)}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

Similarly,

$$u_\downarrow = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

5. Consider a solution to the Dirac equation for massless particles, $u_+(\vec{p})$, where the + denotes the fact that the solution is an eigenstate of the spin operator in the $\hat{p}$ directions,

$$(\vec{\Sigma} \cdot \hat{p}) u_+(\vec{p}, x) = u_+(\vec{p}, x).$$

Show that the operator β operating on $u_+(p)$ gives a negative energy solution but is still an eigenstate of $\vec{\Sigma} \cdot \hat{p}$ with eigenvalue +1.

Solution:

$$\begin{aligned}
\{\beta, \alpha_i\} &= 0, \\
H &= \vec{\alpha} \cdot \vec{p}, \\
\{\beta, H\} &= 0, \\
\Sigma_i &= -\frac{i}{2}\epsilon_{ijk}\alpha_j\alpha_k, \\
\{\beta, \vec{\Sigma} \cdot \vec{p}\} &= p_i\{\beta, \Sigma_i\} = 0 \\
H\beta|\psi\rangle &= -\beta H|\psi\rangle \\
&= -E\beta|\psi\rangle.
\end{aligned}$$

So β switches energy. In contrast,

$$(\vec{\Sigma} \cdot \vec{p})\beta|\psi\rangle = \beta(\vec{\Sigma} \cdot \vec{p})|\psi\rangle.$$

So β leaves $\vec{\Sigma} \cdot \vec{p}$ unchanged, i.e.

$$\begin{aligned}
(\vec{\Sigma} \cdot \vec{p})|\psi\pm\rangle &= \pm|\psi\pm\rangle, \\
(\vec{\Sigma} \cdot \vec{p})\beta|\psi+\rangle &= \beta|\psi+\rangle.
\end{aligned}$$

6. Show that the operator

$$P = \frac{1}{2E_p}(E_p + \vec{\alpha}\cdot\vec{p} + \beta m)$$

(a) is a projector, i.e. $P^2 = P$.

(b) and that $P|\psi\rangle$ gives a positive-energy solution to the Dirac equation when operating on any state $|\psi\rangle$,

$$(E_p - \vec{\alpha}\cdot\vec{p} - \beta m)P|\psi\rangle = 0.$$

Solution:
a) Using the fact that β and α anti-commute and that $\{\alpha_i, \alpha_{j\neq i}\} = 0$ and that $\alpha_i^2 = \beta^2 = \mathbb{I}$,

$$\begin{aligned}
P^2 &= \frac{1}{4E_p^2}\left[E_p^2 + m^2 + |\vec{p}|^2 + 2E_p(\vec{\alpha}\cdot\vec{p} + \beta m)\right] \\
&= \frac{1}{4E_p^2}\left[2E_p^2 + 2E_p(\vec{\alpha}\cdot\vec{p} + \beta m)\right] \\
&= \frac{1}{2E_p}(E_p + \vec{\alpha}\cdot\vec{p} + \beta m) = P.
\end{aligned}$$

b)

$$\begin{aligned}
(E_p - \vec{\alpha}\cdot\vec{p} - \beta m)P|\psi\rangle &= \frac{1}{2E_p}(E_p - \vec{\alpha}\cdot\vec{p} - \beta m)(E_p + \vec{\alpha}\cdot\vec{p} + \beta m)|\psi\rangle \\
&= \frac{1}{2E_p}\left[E_p^2 - m^2 - |\vec{p}|^2\right]|\psi\rangle \\
&= 0.
\end{aligned}$$

7. Consider a massless spin half particle of charge e in a magnetic field in the $\hat{z}$ direction described by the vector potential

$$\vec{A} = Bx\hat{y}.$$

The Hamiltonian is then

$$H = \alpha_x(-i\hbar\partial_x) + \alpha_y(-i\hbar\partial_y - eBx).$$

(a) Show that the Hamiltonian commutes with $-i\hbar\partial_y$ and $-i\hbar\partial_z$.

(b) The wave function can then be written as

$$\psi_{k_y,k_z}(x,y,z) = e^{ik_y y + ik_z z}\phi_{k_y,k_z}(x),$$

After setting $k_y = k_z = 0$, show that the lowest energy can be found by solving the equation

$$E^2\phi_\pm(x) = (-\hbar^2\partial_x^2 + e^2B^2x^2 - e\hbar B\Sigma_z)\phi_\pm(x).$$

(c) Show that the eigenvalues of the operator H^2 are

$$E_\pm^2 = (2n + 1 \mp 1)e\hbar B, \quad n = 0, 1, 2\cdot,$$

where the $\pm$ refers to eigenvalues of Σ_z. You can do this mapping to the harmonic oscillator and then using the solutions to the harmonic oscillator from Chapter 3. Note that when the the eigenvalue of Σ_z is $+1$, there exists a solution with $E = 0$.

Solution:
a) By inspection $-i\hbar\partial_y$ commutes with H because there is no y in H. Same for $-i\hbar\partial_z$.
b)

$$H^2 = -\hbar^2\partial_x^2 + (-i\hbar\partial_y - eBx)^2 - \hbar^2\partial_z^2 - i\hbar\alpha_x\alpha_y eB.$$

Set $-i\hbar\partial_y = \hbar k_y$ and $-i\hbar\partial_z = \hbar k_z$. You can see that answer doesn't depend on k_y, as it just changes position of center of H.O., and one can minimize energy by setting $k_z = 0$, so just set both to zero to find lowest energy, and

$$H^2 = -\hbar^2\partial_x^2 + e^2B^2x^2 - e\hbar B\Sigma_z, \quad \Sigma_z \equiv -i\alpha_x\alpha_y.$$

c) Because Σ_z commutes with H and because $\Sigma_z^2 = \mathbb{I}$, one can see that eigenvalues of Σ_z are ± 1. Thus, by noticing how this looks like a Harmonic Oscillator Hamiltonian (except for $H^2/2M$ rather than for H where M is some fictional quantity which will drop out),

$$\frac{H^2}{2M} = -\frac{\hbar^2}{2M}\partial_x^2 + \frac{1}{2}M\frac{e^2B^2}{M^2}x^2 - \frac{e\hbar B}{2M}\Sigma_z.$$

Looks like H.O. with $\omega = eB/M$, plus a constant addition to the energy of $\mp\frac{e\hbar B}{2M} = \mp\hbar\omega/2$. The energy eigenvalues are thus

$$\frac{E_\pm^2}{2M} = (n + 1/2)\hbar\omega \mp \frac{\hbar\omega}{2},$$
$$E_\pm^2 = ((2n+1) \mp 1)e\hbar B. \checkmark$$

8. Using the definition of field operators in Eq. (**??**), show that the Hamiltonian

$$H = \int d^3r \; \Psi^\dagger(\vec{r},t)(-i\hbar\vec{\alpha}\cdot\nabla + \beta m)\Psi(\vec{r},t)$$
$$= \sum_{s,\vec{p}} E_p(b^\dagger_{s,\vec{p}} b_{s,\vec{p}} + d^\dagger_{s,\vec{p}} d_{s,\vec{p}} - 1).$$

I.e. the vacuum energy for each mode is negative.

Solution:

Using

$$\Psi_i(\vec{r},t) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \sqrt{\frac{m}{E_p}} \sum_s \left(u_{s,i}(\vec{p}) e^{-iE_pt/\hbar + i\vec{p}\cdot\vec{r}/\hbar} b_{s,\vec{p}} + v_{s,i}(\vec{p}) e^{iE_pt/\hbar - i\vec{p}\cdot\vec{r}/\hbar} d^\dagger_{s,\vec{p}} \right).$$

$$H = \frac{m}{V} \int d^3r \; \sum_{\vec{p},s} \sum_{\vec{p}',s'} \frac{1}{\sqrt{E_p E_{p'}}} \left[u^\dagger_s(\vec{p}) u_{s'}(\vec{p}') \right.$$
$$+ u^\dagger_s(\vec{p}) v_{s'}(\vec{p}')$$
$$+ v^\dagger_s(\vec{p}) u_{s'}(\vec{p}')$$
$$\left. + v^\dagger_s(\vec{p}) v_{s'}(\vec{p}') \right].$$

Using the fact that u_p and v_p are eigenstates of the Dirac Hamiltonian with eigenvalues $\pm E_p$ respectively,

$$H = \frac{m}{V} \int d^3r \; \sum_{\vec{p},s} \sum_{\vec{p}',s'} \frac{1}{\sqrt{E_p E_{p'}}} \left[u^\dagger_s(\vec{p}) e^{iE_pt/\hbar - i\vec{p}\vec{r}/\hbar} E_{p'} e^{-iE_{p'}t/\hbar + i\vec{p}'\vec{r}/\hbar} u_{s'}(\vec{p}') \right.$$
$$+ u^\dagger_s(\vec{p}) e^{iE_pt/\hbar - i\vec{p}\vec{r}/\hbar} (-E_{p'}) e^{iE_{p'}t/\hbar - i\vec{p}'\vec{r}/\hbar} v_{s'}(\vec{p}')$$
$$+ v^\dagger_s(\vec{p}) e^{-iE_pt/\hbar + i\vec{p}\vec{r}/\hbar} E_{p'} e^{-iE_{p'}t/\hbar + i\vec{p}'\vec{r}/\hbar} u_{s'}(\vec{p}')$$
$$\left. + v^\dagger_s(\vec{p}) e^{-iE_pt/\hbar + i\vec{p}\vec{r}/\hbar} (-E_{p'}) e^{iE_{p'}t/\hbar - i\vec{p}'\vec{r}/\hbar} v_{s'}(\vec{p}') \right].$$

Using the fact that $\int d^3r e^{i(\vec{p}-\vec{p}')\cdot\vec{r}} = V\delta_{\vec{p}\vec{p}'}$,

$$H = m \sum_{\vec{p},s,s'} \left[u^\dagger_s(\vec{p}) u_{s'}(\vec{p}) b^\dagger_{\vec{p}} b_{\vec{p}} - e^{2iE_pt/\hbar - 2i\vec{p}\cdot\vec{r}/\hbar} u^\dagger_s(\vec{p}) v_{s'}(-\vec{p}) b^\dagger_{\vec{p}} d^\dagger_{\vec{p}}) \right.$$
$$\left. + e^{-2iE_pt/\hbar + 2i\vec{p}\vec{r}/\hbar} v^\dagger_s(\vec{p}) u_{s'}(-\vec{p}) d^\dagger_{\vec{p}} b_{\vec{p}}) - v^\dagger_s(\vec{p}) v_{s'}(\vec{p}) d_{\vec{p}} d^\dagger_{\vec{p}'} \right].$$

Using the orthogonality relations for u and v,

$$H = \sum_{\vec{p},s} E_p(b^\dagger_{\vec{p}} b_{\vec{p}} - d_{\vec{p}} d^\dagger_{\vec{p}})$$
$$= \sum_{\vec{p},s} E_p(b^\dagger_{\vec{p}} b_{\vec{p}} + d^\dagger_{\vec{p}} d_{\vec{p}} - 1).$$

9. Using the definitions for α_k and β_k in Eq. (**??**),

   (a) Show that

$$b^\dagger_{\vec{k}} b_{\vec{k}} - d^\dagger_{-\vec{k}} d_{-\vec{k}} = \alpha^\dagger \alpha_k - \beta^\dagger \beta_k.$$

   This demonstrates that the eigenstates of the new Hamiltonian are still eigenstates of the charge operator written in the old basis.

   (b) Show that the state

$$|\tilde{0}\rangle \equiv \cos\theta_k |0\rangle + \sin\theta_k d^\dagger_{-\vec{k}} b^\dagger_{\vec{k}} |0\rangle$$

   is destroyed by both α_k and β_k, where $|0\rangle$ is the vacuum in the old basis. Effectively, this shows that $|\tilde{0}\rangle$ is the vacuum in the new basis.

Solution:
a) Suppressing the k indices

$$\begin{aligned}
\alpha^\dagger &= \cos\theta b^\dagger + \sin\theta d, \\
\beta^\dagger &= \cos\theta d^\dagger - \sin\theta b, \\
(\alpha^\dagger \alpha - \beta^\dagger \beta &= (\cos\theta b^\dagger + \sin\theta d)(\cos\theta + \sin\theta d^\dagger) - (\cos\theta d^\dagger - \sin\theta b)(\cos\theta d - \sin\theta b^\dagger) \\
&= b^\dagger b \cos^2\theta - b b^\dagger \sin^2\theta + d^\dagger d(-\cos^2\theta) + d d^\dagger \sin^2\theta \\
&+ b^\dagger d^\dagger \cos\theta \sin\theta + d^\dagger b^\dagger \cos\theta \sin\theta + db \sin\theta\cos\theta + bf \sin\theta\cos\theta.
\end{aligned}$$

Using the anti-communtation rules,

$$\begin{aligned}
&= b^\dagger b - \sin^2\theta - d^\dagger d + \sin^2\theta \\
&= b^\dagger b - d^\dagger d.
\end{aligned}$$

b)

$$\begin{aligned}
|\tilde{0}\rangle &= \cos\theta|0\rangle + \sin\theta|d^\dagger b^\dagger|0\rangle, \\
\alpha|\tilde{0}\rangle &= (\cos\theta b + \sin\theta d^\dagger)\{\cos\theta|0\rangle + \sin\theta d^\dagger b^\dagger|0\rangle\} \\
&= \sin\theta\cos\theta d^\dagger + \sin\theta\cos\theta b d^\dagger b^\dagger|0\rangle \\
&= \sin\theta\cos\theta d^\dagger - \sin\theta\cos\theta d^\dagger b b^\dagger|0\rangle \\
&= 0.
\end{aligned}$$