Chapter 12 – Homework Solutions

1. Consider the two electron holes in the p-shell of a neutral oxygen atom.

   (a) What is the $L-S-J$ of the ground state.

   (b) If the atom is in a magnetic field of 0.01 Tesla, find the magnetic energies of the originally degenerate $2J+1$ states.

Solution:
a) Consider 2 holes, $S=0,1$, so $S=1$ is lowest because of Hund's Rule #1.
$L=0,1,2$ From permutation symmetry, $L=0,2$ if orbital WF is to be symmetric and $L=1$ for orbital WF to be anti-symmetric. For spin WFs, $S=0$ is anti-symmetric, while $S=1$ is symmetric. To have $S=1$ and have overall WF being anti-symmetric, one needs $L=1$.
Finally, last Hund's rule prefers highest J, $J=2$.

$$S=1, L=1, J=2.$$

b) From lecture notes,

$$\begin{aligned}\Delta E &= -g\frac{e\hbar B}{2mc}M_J\\ g &= 1+\frac{J(J+1)+S(S+1)-L(L+1)}{2J(J+1)}.\end{aligned}$$

Plugging and chugging,

$$\begin{aligned}\Delta E &= -g\frac{e\hbar B}{2mc}M_J,\\ g &= 1+\frac{6+2-2}{12}=\frac{3}{2},\\ \Delta E &= -\frac{3}{2}\frac{e\hbar}{2mc}BM_J\\ &= -\frac{3}{2}M_J\cdot 5.788\times 10^{-5}\times 0.01\\ &= (8.68\times 10^{-2})M_J \text{ eV}\end{aligned}$$

2. One electron moves in a one-dimensional system and feels the interaction of two atoms. Approximate the interaction between the electrons and the atoms with the potential

$$V(x-R) = -\beta\delta(x-R),$$

where R is the position of an atom. Use the adiabatic approximation to answer the following questions.

(a) Given the two atoms are separated by a distance r, find a transcendental equation relating k and r where the electronic binding energy is $\hbar^2k^2/(2m)$.

(b) Find the potential between the two atoms at small r,

$$V(r \to 0) \sim V(r=0) - \alpha r,$$

that is, find $V(r=0)$ and α. Do this by expanding the transcendental equation in terms of r. Hint: First, find $V(r=0)$ by solving the transcendental equation with $r=0$. Take derivatives of the transcendental equation with respect to r, then solve for dk/dr at $r=0$, and finally find dE/dr to obtain α.

(c) Find the potential between the two atoms at large r,

$$V(r \to \infty) = -\gamma \exp(-2k_\infty r),$$

that is, find γ. Hint: Use first order perturbation theory, assuming the unperturbed wave function is the bound state of one well, and the perturbation is the interaction with the second well.

Solution:
a) place potentials at $x=-r/2$ and $x=r/2$. Define region I as $x<-r/2$, region II as $-r/2<x<r/2$ and $r/2<x$ as region III.

$$\psi_I = Ae^{k(x+r/2)},\ \psi_{II} = \cosh(kx),\quad \psi_{III} = Ae^{-k(x-r/2)},$$

$$BC1)A = \cosh(kr/2),$$

$$BC2) - ka - k\sin(kr/2) = -2m\beta A/\hbar^2,$$

$$-kA + 2m\beta A/\hbar^2 = k\sinh(kr/2),$$

$$-kA + \frac{2m\beta A}{\hbar^2} = k\sinh(kr/2),$$

$$k\tanh(kr/2) = -k + \frac{2m\beta}{\hbar^2},$$

$$\tanh(kr/2) = -1 + \frac{2m\beta}{\hbar^2 k}$$

b)

$$\begin{aligned}
\tanh(kr/2) &= \frac{2m\beta}{\hbar^2 k}, \\
\text{at } r = 0,\ k &= 2m\beta/\hbar^2,\ E = -(2m\beta/\hbar^2)^2 \frac{\hbar^2}{2m}, \\
\frac{d}{dr}\tan(kr/2) &= \frac{d}{dr}\left(\frac{2m\beta}{\hbar^2 k} - 1\right), \\
\frac{k}{2} &= -\frac{2m\beta}{\hbar^2 k^2}\frac{dk}{dr}, \\
\frac{dk}{dr} &= -\frac{m^2\beta^2}{\hbar^4}, \\
E_B &= E(r=0) + \frac{dE}{dk}\frac{dk}{dr} r \\
&= -\frac{2m\beta^2}{\hbar^2} + \frac{\hbar^2 k}{m}\frac{2m^2\beta^2}{\hbar^4} r \\
&= -\frac{2m\beta^2}{\hbar^2} + \frac{2m\beta^2}{\hbar^2} kr \\
&= -\frac{2m\beta^2}{\hbar^2} + \frac{2m\beta^2}{\hbar^2} r \frac{2m\beta}{\hbar^2} \\
&= -\frac{2m\beta^2}{\hbar^2} + 4\frac{m^2\beta^3}{\hbar^4} r.
\end{aligned}$$

c)

$$\begin{aligned}
\psi &\approx \sqrt{k} e^{-k|r|}, \\
k &= \frac{m\beta}{\hbar^2} \quad \text{(single well)} \\
V &= \int \psi^*(r)\psi(r) dr\ (-\beta\delta(r - r_0)) \\
&= -\beta k e^{-2kr_0}, \\
&= -\frac{m\beta^2}{\hbar^2} e^{-2kr_0}.
\end{aligned}$$

3. Consider a particle of mass M and charge e moving in the $x-y$ plane under the influence of a magnetic field in the z direction. Ignore motion in the z direction.

   (a) Show that the vector potential,
   $$\vec{A} = \frac{B}{2}\left(x\hat{y} - y\hat{x}\right),$$
   describes a magnetic field in the z direction.

   (b) Write the Schrödinger equation,
   $$\frac{\left(\vec{P} - e\vec{A}/c\right)^2}{2M}\psi(r,\phi) = E\psi(r,\phi),$$
   in cylindrical coordinates.

   (c) Show that L_z commutes with the Hamiltonian.

   (d) Assuming the solution is an eigenstate of L_z with eigenvalue $m\hbar$,
   $$\psi(r,\phi) = e^{im\phi}\xi_m(r),$$
   rewrite the Schrödinger equation for $\xi_m(r)$.

   (e) **Extra Credit**: Solve for $\xi_m(r)$ and the eigenenergies for the case where $m = 0$.

Solution:

a)

$$\begin{aligned}
(\nabla \times \vec{A})_z &= \partial_x A_y - \partial_y A_x = B/2 + B/2 = B,\\
(\nabla \times \vec{A})_x &= \partial_y A_z - \partial_z A_y = 0,\\
(\nabla \times \vec{A})_y &= \partial_z A_z - \partial_x A_z = 0.
\end{aligned}$$

b)

$$\vec{A} = \frac{B}{2}r\hat{\phi}.$$

c)

$$\begin{aligned}
\nabla &= \hat{z}\partial_z + \hat{r}\partial_r + \frac{\hat{\phi}}{r}\partial_\phi,\\
\nabla^2 &= \partial_z^2 + \partial_r^2 + \frac{1}{r^2}\partial_\phi^2 + \frac{1}{r}\partial_r,\\
E\psi &= -\frac{\hbar^2}{2M}\nabla^2\psi - \frac{ie\hbar}{Mc}\vec{A}\cdot\nabla\psi - \frac{ie\hbar}{2Mc}\nabla\cdot\vec{A}\psi + \frac{e^2}{2Mc^2}|\vec{A}|^2\psi\\
&= -\frac{\hbar^2}{2m}\left(\partial_z^2 + \partial_r^2 + \frac{1}{r}\partial + \frac{1}{r^2}\partial_\phi^2\right)\psi - \frac{ie\hbar B}{2Mc}\partial_\phi\psi + \frac{e^2}{8Mc^2}B^2\psi,
\end{aligned}$$

d)Because $L_z = -i\hbar\partial_\phi$ and there is no ϕ dependence in H, L_z commutes with H.
e)

$$\left\{-\frac{\hbar^2}{2M}\left(\partial_z^2 + \partial_r^2 + \frac{1}{r}\partial + \frac{1}{r^2}\partial_\phi^2\right) + \frac{me\hbar}{2Mc}B + \frac{e^2B^2r^2}{8Mc^2}\right\}\xi_m(r) = E\xi_m(r).$$

f) Set $m = 0$ and guess

$$\xi_m(r) = e^{r^2/2\sigma^2}.$$

Then

$$\begin{aligned}
\partial_r\xi_m(r) &= -\frac{r}{\sigma^2}\xi_m(r), \\
\partial_r^2\xi_m(r) &= \left(-\frac{1}{\sigma^2} + \frac{r^2}{\sigma^4}\right), \\
-\frac{\hbar^2}{2M}\left\{-\frac{1}{\sigma^2} + \frac{r^2}{\sigma^4} - \frac{1}{\sigma^2}\right\} + \frac{e^2B^2r^2}{8Mc^2} &= E, \\
\frac{\hbar^2}{2M\sigma^4} &= \frac{e^2B^2}{8Mc^2}, \\
\sigma^2 &= \sqrt{\frac{4\hbar^2c^2}{e^2B^2}} = \frac{2\hbar c}{eB} \\
E &= \frac{\hbar}{M}\frac{eB}{2c} = \frac{e\hbar B}{2Mc} \\
&= \frac{1}{2}\hbar\omega, \quad \omega = \frac{eB}{Mc}.
\end{aligned}$$

4. Consider a surface with 10 electrons per μm^2. Lowering the magnetic field, at what magnetic field (in Tesla) do you find the first dip in conductivity due to the quantum Hall effect?

Solution:

$$n = 10^13 \text{ cm}^{-2},$$
$$\frac{B}{c} = \frac{2\pi\hbar n}{e} = 0.0414 \text{ Tesla.}$$