Chapter 11 – Homework Solutions

1. Consider a two-dimensional zero-temperature non-relativistic gas of identical spin-up fermions whose mass is m. They are confined in a two-dimensional box of dimensions, L_x and L_y. The quantum numbers characterizing the single-particle eigenstates are n_x and n_y. The box is divided in half along the x axis. The eigenstates with odd n_x now disappear, while there are now two solutions (one for each half of the box) for each even value of n_x. Assuming the size of the box is large compared to the inverse Fermi momentum, find the penalty, expressed as an energy per unit length ($\Delta E/2L_y$), for dividing the box. Express your answer in terms of the Fermi momentum and the mass.

Solution:
For each mode k_y the energy penalty is one half the Fermi energy where the energy from k_y is neglected.

$$\begin{aligned}
\Delta E &= \sum_{k_y,\mathrm{s.t.}\hbar^2\mathrm{k_y^2/2m<\epsilon_f}} \frac{1}{2}\left(\epsilon_f - \frac{\hbar^2 k_y^2}{2m}\right),\\
&= \frac{\hbar^2}{4m}\sum_{k_y<k_f}(k_f^2-k_y^2)\\
&= \frac{\hbar^2}{4m}L_y\int_0^{k_f}\frac{dk_y}{\pi}(k_f^2-k_y^2)\\
&= \frac{\hbar^2}{4m\pi}L_y(k_f^3-k_f^3/3)\\
&= \frac{L_y}{3\pi}k_f\epsilon_f,\\
\frac{\Delta E}{2L_y} &= \frac{k_f\epsilon_f}{6\pi}.
\end{aligned}$$

2. In the interior of a neutron star, the neutron-to-proton ratio is very high. This results despite the fact that the proton mass is 1.3 MeV higher than the neutron mass. This occurs because protons must balanced by an equal number of electrons. Furthermore, protons and neutrons may be interchanged via the reaction,

$$p + e \leftrightarrow n$$

(We have neglected the neutrinos in this reaction because they are free to leave the star due to their massless nature.)

The masses of the particles are:

$$m_pc^2 = 938.27 \text{ MeV}, \quad m_nc^2 = 939.57 \text{ MeV}, \ m_ec^2 = 0.511 \text{ MeV}$$

For the problems below, you may assume the protons and neutrons are non-relativistic, $E = mc^2 + (\hbar ck)^2/(2mc^2)$, but the electrons must be treated as relativistic particles, $E = [(\hbar ck)^2 + (mc^2)^2]^{1/2}$. It is useful to remember that $\hbar c = 197.326$ MeV·fm.

(a) If the baryon(neutrons or protons) density equals n_B, express the corresponding constraint involving the Fermi momenta of the protons and neutrons.

(b) From the constraint that the system is electrically neutral, describe how the proton's Fermi momentum is related to the electron's Fermi momentum.

(c) Describe how minimizing the overall energy results in a constraint involving the three Fermi momenta.

(d) For 4 cases, $n_B = 0.0001, 0.001, 0.1, 1.0$ baryons per cubic Fermi, solve the expressions above and plot the neutron/proton ratio as a function of n_B. Solving the three equations may involve finding roots numerically.

Solution:
a)

$$n_n = \frac{2 \cdot 4\pi}{3} \frac{1}{(2\pi\hbar)^3} p_n^3 = \frac{1}{3\pi^2} \left(\frac{p_n}{\hbar}\right)^3,$$

$$n_p = \frac{1}{3\pi^2} \left(\frac{p_p}{\hbar}\right)^3 = n_e, \text{ thus } p_e = p_p,$$

$$(1) m_p + \frac{p_p^2}{2m} + \sqrt{m_e^2 + p_p^2} = m_n + \frac{p_n^2}{2m_n},$$

$$(2) n_b = \frac{1}{3\pi^2\hbar^3}(p_p^3 + p_n^3)$$

b) $p_p = p_e$ c) Solve the 2 equations and two unknowns (1) and (2).
d) To solve for p_p,

$$m_p - m_n + \frac{p_p^2/2m_p}{+} \sqrt{p_p^2 + m_e^2} = (3\pi^2\hbar^3 n_b - p_p^3)^{2/3}/2m_n.$$

Solving numerically for p_p,

$$p_p* = \begin{cases} 1.65\text{MeV}/c, & n_B = 0.0001\ \text{fm}^{-3} \\ 3.24\text{MeV}/c, & n_B = 0.001\ \text{fm}^{-3} \\ 16.4\text{MeV}/c, & n_B = 0.01\ \text{fm}^{-3} \\ 42.9\text{MeV}/c, & n_B = 0.1\ \text{fm}^{-3} \\ 179\text{MeV}/c, & n_B = 1.0\ \text{fm}^{-3} \end{cases}$$

Applying $p_n^3 = 3\pi^2\hbar^3 n_B - p_p^3$. The ratio of densities is then

$$\frac{n_n}{n_p} = \frac{p_n^3}{p_p^3} = \begin{cases} 5078, & n_B = 0.0001\ \text{fm}^{-3} \\ 6704, & n_B = 0.001\ \text{fm}^{-3} \\ 2003, & n_B = 0.01\ \text{fm}^{-3} \\ 28.6, & n_B = 0.1\ \text{fm}^{-3} \\ 38.6, & n_B = 1.0\ \text{fm}^{-3} \end{cases}$$

3. Calculate the magnetic susceptibility of a two-dimensional electron gas with Fermi wave number k_f.

Solution:

$$
\begin{aligned}
M &= \mu_B(n_\uparrow - n_\downarrow), \\
\epsilon_{f,\uparrow} - \epsilon_{f,\downarrow} &= 2\mu_B B, \\
&= \frac{\hbar^2}{2m}\left(k_{f,\uparrow}^2 - k_{f,\downarrow}^2\right), \\
n &= \frac{1}{(2\pi)^2}\pi\left(k_{f,\uparrow}^2 + k_{f,\downarrow}^2\right), \\
M &= \frac{\mu_B}{(2\pi)^2}\pi\left(k_{f,\uparrow}^2 - k_{f,\downarrow}^2\right) \\
M &= \frac{\mu_B}{4\pi}\frac{2m}{\hbar^2}(2\mu_B B), \\
\chi &= \frac{M}{B} = \frac{\mu_B^2 m}{\hbar^2\pi}.
\end{aligned}
$$

4. Consider the wave function of three identical particles in Eq. (**??**) where the three single-particle wave functions are orthonormalized.

   (a) Show that

$$\frac{1}{3!}\int d^3r_1 d^3r_2 d^3r_3 \; |\phi_{\alpha,\beta,\gamma}(\vec{r}_1,\vec{r}_2,\vec{r}_3)|^2 = 1 \tag{0.1}$$

   if the three indices are different, $\alpha \neq \beta, \alpha \neq \gamma, \beta \neq \gamma$, and that the overlap is zero if any two of the indices are the same.

   (b) Show that $\phi_{\alpha,\beta,\gamma}(\vec{r}_1,\vec{r}_2,\vec{r}_3) = 0$ if any two positions are the same.

Solution:
a) With the indices the same

$$\begin{aligned}
& \phi_\alpha(\vec{r}_1)\phi_\beta(\vec{r}_2)\phi_\gamma(\vec{r}_3) - \phi_\beta(\vec{r}_1)\phi_\alpha(\vec{r}_2)\phi_\gamma(\vec{r}_3) \\
& - \phi_\gamma(\vec{r}_1)\phi_\beta(\vec{r}_2)\phi_\alpha(\vec{r}_3) - \phi_\alpha(\vec{r}_1)\phi_\gamma(\vec{r}_2)\phi_\beta(\vec{r}_3) \\
& + \phi_\beta(\vec{r}_1)\phi_\gamma(\vec{r}_2)\phi_\alpha(\vec{r}_3) + \phi_\gamma(\vec{r}_1)\phi_\alpha(\vec{r}_2)\phi_\beta(\vec{r}_3) \\
\\
\rightarrow \quad & \phi_\alpha(\vec{r}_1)\phi_\alpha(\vec{r}_2)\phi_\alpha(\vec{r}_3) - \phi_\alpha(\vec{r}_1)\phi_\alpha(\vec{r}_2)\phi_\alpha(\vec{r}_3) \\
& - \phi_\alpha(\vec{r}_1)\phi_\alpha(\vec{r}_2)\phi_\alpha(\vec{r}_3) - \phi_\alpha(\vec{r}_1)\phi_\alpha(\vec{r}_2)\phi_\alpha(\vec{r}_3) \\
& + \phi_\alpha(\vec{r}_1)\phi_\alpha(\vec{r}_2)\phi_\alpha(\vec{r}_3) + \phi_\alpha(\vec{r}_1)\phi_\alpha(\vec{r}_2)\phi_\alpha(\vec{r}_3) \\
\\
& = 0
\end{aligned}$$

b) With $\vec{r}_1 = \vec{r}_2$,

$$\begin{aligned}
& \phi_\alpha(\vec{r}_1)\phi_\beta(\vec{r}_2)\phi_\gamma(\vec{r}_3) - \phi_\beta(\vec{r}_1)\phi_\alpha(\vec{r}_2)\phi_\gamma(\vec{r}_3) \\
& - \phi_\gamma(\vec{r}_1)\phi_\beta(\vec{r}_2)\phi_\alpha(\vec{r}_3) - \phi_\alpha(\vec{r}_1)\phi_\gamma(\vec{r}_2)\phi_\beta(\vec{r}_3) \\
& + \phi_\beta(\vec{r}_1)\phi_\gamma(\vec{r}_2)\phi_\alpha(\vec{r}_3) + \phi_\gamma(\vec{r}_1)\phi_\alpha(\vec{r}_2)\phi_\beta(\vec{r}_3) \\
\\
\rightarrow \quad & \phi_\alpha(\vec{r}_1)\phi_\beta(\vec{r}_1)\phi_\gamma(\vec{r}_3) - \phi_\beta(\vec{r}_1)\phi_\alpha(\vec{r}_1)\phi_\gamma(\vec{r}_3) \\
& - \phi_\gamma(\vec{r}_1)\phi_\beta(\vec{r}_1)\phi_\alpha(\vec{r}_3) - \phi_\alpha(\vec{r}_1)\phi_\gamma(\vec{r}_1)\phi_\beta(\vec{r}_3) \\
& + \phi_\beta(\vec{r}_1)\phi_\gamma(\vec{r}_1)\phi_\alpha(\vec{r}_3) + \phi_\gamma(\vec{r}_1)\phi_\alpha(\vec{r}_1)\phi_\beta(\vec{r}_3) \\
\\
& = 0
\end{aligned}$$

5. Correlation/anti-correlation in a quantum gas: Consider a uniform gas of non-interacting spin-half particles in the ground state. The wave function may be written

$$|\phi\rangle = \prod_{\alpha, |k_\alpha| < k_f} a_\alpha^\dagger |0\rangle,$$

where the product includes all states α with momentum $k_\alpha < k_f$ and spin s_α. The density-density correlation function is defined as

$$g_{s_1,s_2}(\vec{x}_2 - \vec{x}_1) \equiv \frac{\langle\phi|\Psi_{s_1}^\dagger(\vec{x}_1)\Psi_{s_2}^\dagger(\vec{x}_2)\Psi_{s_2}(\vec{x}_2)\Psi_{s_1}(\vec{x}_1)|\phi\rangle}{\langle\phi|\Psi_{s_1}^\dagger(\vec{x}_1)\Psi_{s_1}(\vec{x}_1)|\phi\rangle\langle\phi|\Psi_{s_2}^\dagger(\vec{x}_2)\Psi_{s_2}(\vec{x}_2)|\phi\rangle}$$

This function expresses the correlation of two particles with spin s_1 and s_2 being separated by $\vec{x}_2 - \vec{x}_1$. It is defined in such a way that it is unity if the probability of seeing two particles at $\vec{x}_1$ and $\vec{x}_2$ is the product of the probabilities of observing each particle independently.

(a) Show that the density-density correlation function can be written as

$$g_{s_1,s_2}(\vec{x}_2 - \vec{x}_1) = 1 - \delta_{s_1 s_2} \frac{\sum_{k_\alpha, k_\beta} e^{i(\vec{k}_\alpha - \vec{k}_\beta)\cdot(\vec{x}_2 - \vec{x}_1)}}{\sum_{k_\alpha, k_\beta}},$$

where the sums are over all momentum states with $k_\alpha < k_f$ and $k_\beta < k_f$.

(b) By changing the sum over states to a three-dimensional integral over $\vec{k}$, find an analytic expression for the density-density correlation function in terms of the Fermi momentum k_f.

Solution:
a)

$$\Psi_{s_2}(x_2)\Psi_{s_1}(x_1)\prod_\gamma a_\gamma^\dagger|0\rangle = \sum_{\alpha,\beta,k_\alpha<k_f,k_\beta<k_f} e^{ik_\alpha x_1 + ik_\beta x_2}\delta_{s_1 s_\alpha}\delta_{s_2 s_\beta}\left(\prod_{\gamma\neq\alpha,\beta,k_\gamma<k_f} a_\gamma^\dagger|0\rangle\right)$$

$$\langle 0|\left(\prod_\gamma a_\gamma\right)\Psi_{s_1}^\dagger(x_1)\Psi_{s_2}^\dagger(x_2) = \sum_{\alpha'\beta'} e^{-ik_{\alpha'}x_1 - ik_{\beta'}x_2}\langle 0|\left(\prod_{\gamma\neq\alpha',\beta',\;\; k_{\gamma'}<k_f} a_\gamma\right)\delta_{s_1 s_{\alpha'}}\delta_{s_2 s'_\beta}$$

$$\begin{aligned}\langle\phi|\Psi_{s_1}^\dagger(x_1)\Psi_{s_2}^\dagger(x_2)\Psi_{s_2}(x_2)\Psi_{s_1}(x_1)|\phi\rangle &= \sum_{\alpha,\beta,k_\alpha<k_f,k_\beta<k_f}\left[e^{ik_\alpha x_1 + ik_\beta x_2 - ik_\alpha x_1 - ik_\beta x_2}(\delta_{s_1 s_\alpha}\delta_{s_2 s_\beta})^2\right.\\ &\left.- e^{ik_\alpha x_1 + ik_\beta x_2 - ik_\alpha x_2 - ik_\beta x_1}\delta_{s_1 s_\alpha}\delta_{s_2 s_\beta}\delta_{s_2 s_\alpha}\delta_{s_1 s_\beta}\right]\\ &= \sum_{\alpha,\beta,k_\alpha<k_f,k_\beta<k_f} - \delta_{s_1 s_2}\sum_{\alpha,\beta,k_\alpha<k_f,k_\beta<k_f} e^{i(k_\alpha - k_\beta)(x_2 - x_1)}\end{aligned}$$

$$C_{s_1 s_2}(x_2 - x_1) = 1 - \delta_{s_1 s_2}\frac{\sum_{k_\alpha<k_f,k_\beta<k_f} e^{i(k_\alpha - k_\beta)(x_2 - x_1)}}{\sum_{k_\alpha<k_f,k_\beta<k_f}}.$$

b)

$$\sum_{\alpha} \rightarrow \frac{V}{(2\pi)^3} \int_{k_\alpha < k_f} d^3 k_\alpha,$$

$$\begin{aligned}
C_{s_1 s_2}(x_2 - x_1) &= 1 - \delta_{s_1 s_2} \left(\frac{4\pi}{3} k_f^3\right)^{-2} I^2, \\
I^2(\Delta x) &\equiv \int d^3 k_\alpha d^3 k_\beta \; e^{i(k_\alpha - k_\beta)(\Delta x)} \\
I &= \int_{k<k_f} d^3 k \cos(\vec{k} \cdot \Delta \vec{x}) \;\; \text{(depends only on } |\Delta \vec{x}| \text{ so pick along z} - \text{axis)} \\
&= 4\pi \int_{k_\perp < k_f} k_\perp dk_\perp \; \int_0^{(k_f^2 - k_\perp^2)^{1/2}} dk_z \cos(k_z \delta x) \\
&= 4\pi \int_0^{k_f} k_\perp dk_\perp \frac{1}{\Delta x} \sin(\sqrt{k_f^2 - k_\perp^2} \Delta x) \\
u &\equiv \sqrt{k_f^2 - k_\perp^2}, \quad du = \frac{-1}{u} k_\perp dk_\perp, \\
&= 4\pi \int_0^{k_f} \int_0^{k_f} du \; \frac{u du}{\Delta x} \sin(u \Delta x), \\
&= \frac{4\pi}{\Delta x^3} \sin(k_f \Delta x) - \frac{4\pi}{\Delta x^2} k_f \cos(k_f \Delta x) \;\; \text{integrated by parts twice} \\
C_{s_1, s_2}(\Delta x) &= 1 - \delta_{s_1 s_2} \left[\frac{\sin(k_f \Delta x)/\Delta x^3 - k_f \cos(k_f \Delta x)/\Delta x^2}{k_f^3/3}\right]^2.
\end{aligned}$$

If you do a Taylor expansion in small Δx of both the sin and cos terms, you will see that $C_{s_1 = s_2}(\Delta x = 0) = 0$

6. Again, consider a non-interacting quantum gas of particles of mass m, with the ground state being expressed as

$$|\phi\rangle = \prod_{\alpha, |k_\alpha| < k_f} a_\alpha^\dagger |0\rangle,$$

where the product includes all states α with momentum $k_\alpha < k_f$. For this problem, ignore the spin indices (as if only spin $\uparrow$ particles existed. Consider an interaction between the particles of the form,

$$\begin{aligned} H_{\rm int} &= \frac{1}{2}\int d^3r_1 d^3r_2 V(\vec{r}_1 - \vec{r}_2)\Psi^\dagger(\vec{r}_1)\Psi^\dagger(\vec{r}_2)\Psi(\vec{r}_2)\Psi(\vec{r}_1) \\ V(\vec{r}_1 - \vec{r}_2) &= \beta\delta(\vec{r}_2 - \vec{r}_1) \end{aligned}$$

Find the first-order perturbative correction for the energy, $\langle\phi|H_{\rm int}|\phi\rangle$, for particles in the gas in terms of k_f, β and m. Use the result from the previous problem assuming only one spin exists,

$$C(\Delta x) = 1 - \left[\frac{\sin(k_f\Delta x)/\Delta x^3 - k_f\cos(k_f\Delta x)/\Delta x^2}{k_f^3/3}\right]^2.$$

Solution:

$$\begin{aligned} \langle H_{\rm int}\rangle &= \frac{1}{2}\int d^3r_1 d^3r_2\ V(\vec{r})n^2 C(\vec{r}_1 - \vec{r}_2) \\ n &= \frac{1}{(2\pi)^3}\frac{4\pi}{3}k_f^3 = \frac{k_f^3}{6\pi^2}, \\ \langle H_{\rm int}\rangle &= \frac{V}{2}\left(\frac{k_f^3}{6\pi^2}\right)^2 \int d^3r\ V(\vec{r})\left\{1 - \left[\frac{\sin(k_f r)/r^3 - k_f\cos(k_f r)/r^2}{k_f^3/3}\right]^2\right\} \qquad = 0 \end{aligned}$$

because $V \sim \delta(\vec{r})$ and $C(r=0) = 0$.