Chapter 10 – Homework Solutions

1. The $\Delta^{++,+,0,-}$ baryons have isospin 3/2 while the $\pi^{+,0,-}$ mesons form an isotriplet. Calculate the branching ratios of all four Δ decays into the corresponding $p\pi$ or $n\pi$ channels. (For instance, what fraction of the Δ^+s decay into $p\pi^0$ vs the $n\pi^+$ channels.)

Solution:

$$|I = 3/2, I + z = 3/2\rangle = |I_p = 1/2, I_\pi = 1\rangle$$
$$\sqrt{(3/2)(5/2) - (3/2)(1/2)}|I = 3/2\rangle, I_z = 1/2\rangle = \sqrt{(1/2)(3/2) + (1/2)(1/2)}|I_p = -1/2, I_p = 1\rangle$$
$$+ \sqrt{2}|I_p = 1/2, I_\pi = 0\rangle$$
$$|I = 3/2\rangle, I_z = 1/2\rangle = \frac{1}{\sqrt{3}}\left\{|I_p = -1/2, I_\pi = 1\rangle + \sqrt{2}|I_p = 1/2, I_\pi = 0\rangle\right\},$$
$$|I = 3/2, I_z = -1/2\rangle = \frac{1}{\sqrt{3}}\left\{|I_p = 1/2, I_\pi = -1\rangle + \sqrt{2}|I_p = -1/2, I_\pi = 0\rangle\right\},$$
$$|I = 3/2, I_z = -3/2\rangle = |I_p = -1/2, I_\pi = -3/2\rangle.$$

Thus,
$\Delta^{++} \to p\pi^+$ (100%)
$\Delta^+ \to n\pi^+$ (33%), $\Delta^+ \to p\pi^-$ (67%),
$\Delta^0 \to p\pi^-$ (33%), $\Delta^0 \to n\pi^0$ (67%),
$\Delta^- \to n\pi^-$ (100%)

2. The $S(975)$ meson is an isoscalar ($I = 0$), and decays into two pions. What fraction of the two-pion decays are expected to go into the neutral pion channel?

Solution:

$$
\begin{aligned}
|I=2, I+z=2\rangle &= |\pi^+\pi^+\rangle \\
\sqrt{2\cdot 3-2}|I=2, I_z=1\rangle &= \sqrt{2}|\pi^0\pi^+\rangle + \sqrt{2}|\pi^+\pi^0\rangle, \\
|I=2, I_z=1\rangle &= \frac{1}{\sqrt{2}}(|\pi^0\pi^+\rangle + |\pi^+\pi^0\rangle), \\
\sqrt{6}|I=1, I_z=0\rangle &= \frac{1}{\sqrt{2}}\left\{\sqrt{2}|\pi^0\pi^0\rangle + \sqrt{2}|\pi^+\pi^-\rangle + \sqrt{2}|\pi^0\pi^0\rangle + \sqrt{2}|\pi^-\pi^+\rangle\right\}, \\
|I=2, I_z=0\rangle &= \frac{1}{\sqrt{6}}\left\{2|\pi^0\pi^0\rangle + |\pi^+\pi^-\rangle + |\pi^-\pi^+\rangle\right\}
\end{aligned}
$$

No, onto the $I = 1$ states. Using orthogonality,

$$
\begin{aligned}
|I=1, I_z=1\rangle &= \frac{1}{\sqrt{2}}\left\{|\pi^+\pi^0\rangle - |\pi^0\pi^+\rangle\right\}, \\
\sqrt{2}|I=1, I_z=0\rangle &= \frac{1}{\sqrt{2}}\left\{\sqrt{2}|\pi^0\pi^0\rangle + \sqrt{2}|\pi^+\pi^-\rangle - \sqrt{2}|\pi^-\pi^+\rangle - \sqrt{2}|\pi^0\pi^0\rangle\right\}
\end{aligned}
$$

To get the $I = 0$ state we use orthogonality again. By inspection,

$$
|I=0, I_z=0\rangle = \frac{1}{\sqrt{3}}\left\{|\pi^+\pi^-\rangle + |\pi^-\pi^+\rangle + |\pi^0\pi^0\rangle\right\}.
$$

The the $S(975)$ decays 2/3 of the time to charged pions and 1/3 of the time to neutral pions.

3. Write the Racah coefficient, $W(j_1, j_2, j_3, J; J_{12}, J_{23})$ which is defined by

$$\begin{aligned}&\langle (j_1, j_2), J_{12}, j_3, J, M' | j_1, (j_2, j_3), J_{23}, J, M\rangle \\ &\qquad = \delta_{M,M'} \sqrt{(2J_{12}+1)(2J_{23}+1)} W(j_1, j_2, j_3, J; J_{12}, J_{23}),\end{aligned}$$

in terms of Clebsch-Gordan coefficients.

Solution:

$$\begin{aligned}
|J_{23}, J, M\rangle &= \sum_{M_{23}, M_1} \langle JM | J_{23} M_{23} J_1 M_1\rangle |J_{23} M_{23} M_1\rangle \\
&= \sum_{M_{23}, M_1} \langle JM | J_{23} M_{23} J_1 M_1\rangle \sum_{M_2 M_3} \langle J_{23} M_{23} | J_2 M_2 J_3 M_3\rangle |M_1, M_2, M_3\rangle, \\
|J_{12}, J, M\rangle &= \sum_{M_{12}, M_3} \langle JM | J_{12} M_{12} J_3 M_3\rangle |J_{12}, M_{12}, M_3\rangle \\
&= \sum_{M_{12} M_3} \langle JM | J_{12} M_{12} J_3 M_3\rangle \sum_{M_1 M_2} \langle J_{12} M_{12} | J_1 M_1 J_2 M_2\rangle |M_1, M_2, M_3\rangle, \\
\langle J_{12}, J, M' | J_{23}, J, M\rangle &= \delta_{MM'} \sum_{M_1, M_2, M_3, M_{12}, M_{23}} \langle JM | J_{23} M_{23} JM\rangle \langle J_{23} M_{23} | J_2 M_2 J_3 M_3\rangle \\
&\qquad \langle JM | J_{12} M_{12} J_3 M_3\rangle \langle J_{12} M_{12} | J_1 M_1 J_2 M_2\rangle.
\end{aligned}$$

4. For each operator, define a set (or sets) of irreducible tensor operators T_q^k, from which one can then define the given operator as a linear sum of the irreducible operators. (When defining a set, write down T_q^k for all possible q.)

   (a) z
   (b) p_x
   (c) x^2
   (d) $L_x L_y$

Solution:
From the lecture notes,

$$\begin{aligned}
1 &= T_0^0, \\
x &= \frac{1}{\sqrt{2}}(T_{-1}^1 - T_1^1), \\
y &= \frac{i}{\sqrt{2}}(T_{-1}^1 + T_1^1), \\
z &= T_0^1, \\
x^2 &= \frac{1}{2}\sqrt{\frac{2}{3}}(T_2^2 + T_{-2}^2) - \frac{1}{3}T_0^2 + \frac{1}{3}T_0^0 r^2, \\
y^2 &= -\frac{1}{2}\sqrt{\frac{2}{3}}(T_2^2 + T_{-2}^2) - \frac{1}{3}T_0^2 + \frac{1}{3}T_0^0 r^2, \\
z^2 &= \frac{2}{3}T_0^2 + \frac{1}{3}T_0^0 r^2, \\
xy &= i\frac{1}{\sqrt{6}}(T_{-2}^2 - T_2^2), \\
xz &= \frac{1}{\sqrt{3}}(T_{-1}^2 - T_1^2), \\
yz &= \frac{i}{\sqrt{3}}(T_{-1}^2 + T_1^2).
\end{aligned}$$

a) $z = T^{10}$, where

$$T_1^1 = -\frac{1}{\sqrt{2}}(x + iy), \quad T_{-1}^1 = \frac{1}{\sqrt{2}}(x + iy), \quad T_0^1 = z.$$

b) $p_x = (T_{-1}^1 - T_1^1)/\sqrt{2}$, where

$$T_1^1 = -\frac{1}{\sqrt{2}}(p_x + ip_y), \quad T_{-1}^1 = \frac{1}{\sqrt{2}}(p_x + ip_y), \quad T_0^1 = p_z.$$

c) $x^2 = (T_2^2 + T_{-2}^2)/\sqrt{6} - T_0^2/3 + T_0^0/3$, where

$$\begin{aligned}
T_{\pm 2}^2 &= (x + iy)^2, \\
T_{\pm 1}^2 &= \pm 2z(x \pm iy), \\
T_0^2 &= (2z^2 - x^2 - y^2)\sqrt{2}, \\
T_0^0 &= 1.
\end{aligned}$$

d) $L_x L_y = i(T^2_{-2} - T^2_2)/\sqrt{6}$, where

$$
\begin{aligned}
T^2_{\pm 2} &= (L_x + iL_y)^2, \\
T^2_{\pm 1} &= \pm 2L_z(L_x \pm iL_y), \\
T^2_0 &= (2L_z^2 - L_x^2 - L_y^2)\sqrt{2},
\end{aligned}
$$

5. In terms of the Pauli matrices, find the rotation matrix $\mathcal{D}^{(j)}_{mm'}(\phi, \theta, \psi)$ for the case where $j = 1/2$, and ϕ, θ and ψ are Euler angles.

Solution:

$$
\begin{aligned}
J_z/\hbar &= \sigma_z/2, \quad J_y/\hbar = \sigma_y/2, \\
R(\phi,\theta,\psi) &= e^{-i\sigma_z\phi/2}e^{-i\sigma_y\theta/2}e^{-i\sigma_z\psi/2} \\
&= [\cos(\phi/2) - i\sigma_z\sin(\phi/2)][\cos(\theta/2) - i\sigma_y\sin(\theta/2)][\cos(\psi/2) - i\sigma_z\sin(\psi/2)] \\
&= \cos(\phi/2)\cos(\theta/2)\cos(\psi/2) - \sin(\phi/2)\cos(\theta/2)\sin(\psi/2) \\
&\quad - i\sigma_z[\sin(\phi/2)\cos(\theta/2)\cos(\psi/2) + \cos(\phi/2)\cos(\theta/2)\sin(\psi/2)] \\
&\quad - i\sigma_x[\cos(\phi/2)\sin(\theta/2)\sin(\psi/2) - \sin(\phi/2)\sin(\theta/2)\cos(\psi/2)] \\
&\quad - i\sigma_y[\sin(\phi/2)\sin(\theta/2)\sin(\psi/2) + \cos(\phi/2)\sin(\theta/2)\cos(\psi/2)] \\
&= \cos(\theta/2)\cos((\phi+\psi)/2) - i\sigma_z\cos(\theta/2)\sin((\phi+\psi)/2) \\
&\quad - i\sigma_x\sin(\theta/2)\sin((\psi-\phi)/2) - i\sigma_y\sin(\theta/2)\cos((\psi-\phi)/2).
\end{aligned}
$$

6. Using Eq.s (**??**) and (**??**) derive Eq. (**??**).

Solution:

$$\begin{aligned}
R &= e^{-iJ_{z'}\psi/\hbar}e^{-iJ_{y'}\theta/\hbar}e^{-iJ_z\phi/\hbar}, \\
e^{-iJ_{z'}\psi/\hbar} &= e^{-iJ_{y'}\theta/\hbar}e^{-iJ_z\psi/\hbar}e^{iJ_{y'}\theta/\hbar}, \\
R &= e^{-iJ_{y'}\theta/\hbar}e^{-iJ_z\psi/\hbar}e^{iJ_{y'}\theta/\hbar}e^{-iJ_{y'}\theta/\hbar}e^{-iJ_z\phi/\hbar} \\
&= e^{-iJ_{y'}\theta/\hbar}e^{-iJ_z\psi/\hbar}e^{-iJ_z\phi/\hbar}, \\
e^{-iJ_{y'}\theta/\hbar} &= e^{-iJ_z\phi/\hbar}e^{-iJ_y\theta/\hbar}e^{iJ_z\phi/\hbar}, \\
R &= e^{-iJ_z\phi/\hbar}e^{-iJ_y\theta/\hbar}e^{J_z\phi/\hbar}e^{-iJ_z\psi/\hbar}e^{-iJ_z\phi/\hbar} \\
&= e^{-iJ_z\phi/\hbar}e^{-iJ_y\theta/\hbar}e^{-iJ_z\psi/\hbar} \qquad \checkmark
\end{aligned}$$

7. Circle the matrix elements that might be non-zero.

- $\langle \alpha', J' = 2, M' = 1 | P_x^2 + P_y^2 | \alpha, J = 4, M = 3 \rangle$
- $\langle \alpha', J' = 2, M' = 1 | P_x P_y | \alpha, J = 4, M = 1 \rangle$
- $\langle \alpha', J' = 2, M' = 1 | \epsilon_{ijk} J_i R_j P_k | \alpha, J = 2, M = 1 \rangle$
- $\langle \alpha', J' = 3, M' = 1 | \epsilon_{ijk} J_i R_j P_k | \alpha, J = 2, M = 1 \rangle$
- $\langle \alpha', J' = 2, M' = 1 | P_x | \alpha, J = 3, M = 1 \rangle$
- $\langle \alpha', J' = 2, M' = 1 | P_x | \alpha, J = 3, M = 0 \rangle$
- $\langle \alpha', J' = 2, M' = 1 | P_x | \alpha, J = 2, M = 0 \rangle$

Solution:

- $\langle \alpha', J' = 2, M' = 1 | P_x^2 + P_y^2 | \alpha, J = 4, M = 3 \rangle$
- $\langle \alpha', J' = 2, M' = 1 | P_x P_y | \alpha, J = 4, M = 1 \rangle$
- $\langle \alpha', J' = 2, M' = 1 | \epsilon_{ijk} J_i R_j P_k | \alpha, J = 2, M = 1 \rangle$ ✓
- $\langle \alpha', J' = 3, M' = 1 | \epsilon_{ijk} J_i R_j P_k | \alpha, J = 2, M = 1 \rangle$
- $\langle \alpha', J' = 2, M' = 1 | P_x | \alpha, J = 3, M = 1 \rangle$
- $\langle \alpha', J' = 2, M' = 1 | P_x | \alpha, J = 3, M = 0 \rangle$✓
- $\langle \alpha', J' = 2, M' = 1 | P_x | \alpha, J = 2, M = 0 \rangle$✓

8. Assume one is calculating matrix elements for transitions from a d state to a s state via a quadrupole type coupling, and that one has performed an integral and found

$$I \equiv \langle \ell' = 0, m' = 0|(z^2 - r^2/3)|\ell = 2, m = 0\rangle.$$

Given that one knows I, find

$$\langle \ell' = 0, m' = 0|(x^2 - r^2/3)|\ell = 2, m\rangle$$

for all five values of m, in terms of I. You can leave your answer in terms of Clebsch-Gordan coefficients.

Solution:

$$\begin{aligned}
z^2 &= (2/3)T^2_0 + (1/3)T^0_0,\\
I &= \langle \ell' = 0, m' = 0|2T^2_0/3|\ell = 2, m = 0\rangle,\\
x^2 &= \sqrt{\frac{1}{6}}(T^2_2 + T^2_{-2}) - \frac{1}{3}T^2_0 + \frac{1}{3}T^0_0,\\
\langle \ell' = 0, m' = 0|x^2|\ell = 2, m\rangle &= \frac{1}{\sqrt{6}}\langle \ell', m = 0|T^2_2 + T^2_{-2}|\ell = 2, m\rangle - \frac{1}{3}\langle \ell' = 0, m' = 0|T^2_0|\ell = 2, m\rangle\\
&= \frac{3I/2}{\langle 00|2, 0, 2, 0\rangle}\left\{\frac{1}{\sqrt{6}}(\langle 0, 0|2, 2, 2, m\rangle + \langle 0, 0|2, 2, -2, m\rangle) - \frac{1}{3}\langle 0, 0|2, 2, 0, m\rangle\right.\\
&= \frac{3I/2}{\langle 00|2, 0, 2, 0\rangle}\left\{\begin{array}{rl}
\langle 0, 0|2, 2, -2, m\rangle/\sqrt{6}, & m = 2\\
\langle 0, 0|2, 2, 2, m\rangle/\sqrt{6}, & m = -2\\
\langle 0, 0|2, 2, 0, m\rangle, & , m = 0\\
0, & m = \pm 1
\end{array}\right.
\end{aligned}$$

9. The matrix element for the electromagnetic decay of an atomic d state with m_i to a p state with m_f is given by the matrix element,

$$\mathcal{M} \equiv \alpha \vec{\epsilon}^* \cdot \langle \ell = 1, m_f | \vec{r} | \ell = 2, m_i \rangle$$

where $\alpha \vec{\epsilon}^* \cdot \vec{r}$ is the interaction responsible for the decay, and $\vec{\epsilon}$ is the polarization vector of the outgoing photon.

Consider the intensity of RCP light that is emitted along the z axis. The polarization vector of such light can be written as $(1/\sqrt{2})(\hat{x} + i\hat{y})$. Find the RELATIVE intensities of such light for all 15 combinations of m_i and m_f. You can get Clebsch-Gordan coefficients from a table, e.g. https://en.wikipedia.org/wiki/Table_of_Clebsch-Gordan_coefficients, or use an on-line calculator, e.g. https://www.wolframalpha.com/input/?i=Clebsch-Gordan+calculator.

Solution:

$$\begin{aligned} \vec{\epsilon}^* \cdot \vec{r} &\sim (x - iy) \sim T^1_{-1}, \\ \text{Intensity} &\sim |\langle \ell = 1, m_f | T^1_{-1} | \ell = 2, m_i \rangle|^2 \\ &\sim |\langle 1, m_f | 1, -1, 2, m_i \rangle|^2. \end{aligned}$$

Looking up the C.G. coefficients the intensities are proportional to

$$\text{Intensity} \sim \begin{cases} 3/5, & m_i = 2, m_f = 1 \\ 3/10, & m_i = 1, m_f = 0 \\ 1/10, & m_i = 0, m_f = -1 \\ 0, & \text{otherwise} \end{cases}$$