your name(s)____

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Physics 852 Exercise #4c - Friday, Feb. 11th

The radial hydrogen-atom wave functions are

$$\psi_{n,\ell}(r,\theta,\phi) = \left\{ \left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3} \right\}^{1/2} e^{-r/(na_0)} \left(\frac{2r}{na_0}\right)^\ell L_{n+\ell}^{2\ell+1}\left(\frac{2r}{na_0}\right) Y_{\ell,m}(\theta,\phi).$$

Here, $L_n^k(x)$ are Laguerre polynomials. The states of the hydrogen atom are denoted $|n,\ell,J,M\rangle$, where ℓ is the orbital quantum number and $\vec{J}$, is the total angular momentum, $\vec{J} = \vec{L} + \vec{S}$, and M labels the projection of $\vec{J}$. Note that because $s = 1/2$, J must be half-integer.

Consider the operator

$$A \equiv \vec{S} \cdot \vec{P},$$

where R_0 is some arbitrary constant. The operators X, Y, Z are the position operators and $R^2 = X^2 + Y^2 + Z^2$.

1. For a given J, what values of ℓ are possible?

2. Write the operator A in terms of a sum over irreducible tensor operators, T_q^k, where you define the operators. (You can find this in the lecture notes or peak at the FYI below).

3. You need to calculate the matrix elements

$$\langle n', \ell', J', M'|A|n,\ell,J,M\rangle.$$

   For a given ket state, $|n,\ell,J,M\rangle$, which values of n', ℓ', J', M' might result in a non-zero matrix element? Use the Wigner-Eckart theorem along with parity arguments.

4. (EXTRA CREDIT) Repeat, but with the operator A being replaced by

$$B \equiv S_z P_z.$$

FYI: Some spherical harmonics are:

$$
\begin{aligned}
Y_{0,0} &= \frac{1}{\sqrt{4\pi}}, \\
Y_{1,0} &= \sqrt{\frac{3}{4\pi}} \cos\theta, \\
Y_{1,\pm 1} &= \mp\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}, \\
Y_{2,0} &= \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1), \\
Y_{2,\pm 1} &= \mp\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{\pm i\phi}, \\
Y_{2,\pm 2} &= \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{\pm 2i\phi}, \\
Y_{\ell -m}(\theta,\phi) &= (-1)^m Y^*_{\ell m}(\theta,\phi).
\end{aligned}
$$

An example of some sets of irreducible tensor operators:

$$
\begin{aligned}
T^0_0 &= 1, \\
T^1_1 &= -\frac{1}{\sqrt{2}}(x+iy), \\
T^1_0 &= z, \\
T^1_{-1} &= \frac{1}{\sqrt{2}}(x-iy), \\
T^2_2 &= \sqrt{\frac{3}{8}}(x^2+2ixy-y^2), \\
T^2_1 &= -\frac{\sqrt{3}}{2}z(x+iy), \\
T^2_0 &= \frac{1}{2}(3z^2-r^2), \\
T^2_{-1} &= \frac{\sqrt{3}}{2}z(x-iy), \\
T^2_{-2} &= \sqrt{\frac{3}{8}}(x^2-2ixy-y^2),
\end{aligned}
$$

$$
\begin{aligned}
1 &= T^0_0, \\
x &= \frac{1}{\sqrt{2}}(T^1_{-1} - T^1_1), \\
y &= \frac{i}{\sqrt{2}}(T^1_{-1} + T^1_1), \\
z &= T^1_0, \\
x^2 &= \frac{1}{2}\sqrt{\frac{2}{3}}(T^2_2 + T^2_{-2}) - \frac{1}{3}T^2_0 + \frac{1}{3}T^0_0 r^2, \\
y^2 &= -\frac{1}{2}\sqrt{\frac{2}{3}}(T^2_2 + T^2_{-2}) - \frac{1}{3}T^2_0 + \frac{1}{3}T^0_0 r^2, \\
z^2 &= \frac{2}{3}T^2_0 + \frac{1}{3}T^0_0 r^2, \\
xy &= i\frac{1}{\sqrt{6}}(T^2_{-2} - T^2_2), \\
xz &= \frac{1}{\sqrt{3}}(T^2_{-1} - T^2_1), \\
yz &= \frac{i}{\sqrt{3}}(T^2_{-1} + T^2_1).
\end{aligned}
$$