your name(s)________________________________

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Physics 851 Exercise #6

Continuing from Chapter 4, number 3 in the homework: You considered the matrices

$$\boldsymbol{S_x} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \boldsymbol{S_y} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \boldsymbol{S_z} = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

1. Write the matrices

$$\boldsymbol{S_\pm} = \boldsymbol{S_x} \pm i\boldsymbol{S_y}.$$

   Show what happens when $\boldsymbol{S_\pm}$ acts on each of the three basis vectors.

2. Write down the matrix that performs a rotation by an angle ϕ about the z axis, $\boldsymbol{R_z}(\phi) = e^{-i\boldsymbol{S_z}\phi/\hbar}$.

3. In this basis, we will define $\hat{x}, \hat{y}$ and $\hat{z}$ in terms of the basis vectors,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{y}),$$
$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \hat{z}$$
$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y}).$$

   Express $\hat{x}$, $\hat{y}$ and $\hat{z}$ as vectors.

4. Apply the rotation matrix, $\boldsymbol{R_z}(\phi)$, on $\hat{x}$, then express the result in terms of $\hat{x}$, $\hat{y}$ and $\hat{z}$.

5. Find the matrix $\boldsymbol{U}$ that transforms the coordinate system into one where

$$\hat{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

6. Transform $\boldsymbol{R_z}(\phi)$ to this new basis, i.e. find $\boldsymbol{R'_z} = \boldsymbol{U}\boldsymbol{R_z}\boldsymbol{U}^\dagger$.

Solution:

1)

$$S_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

2)

$$R_z(\phi) = e^{iS_z\phi/\hbar} = \begin{pmatrix} e^{i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\phi} \end{pmatrix}.$$

3)

$$\hat{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\hat{y} = \frac{-i}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\hat{z} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

4)

$$\begin{aligned} R_z(\phi)\hat{x} &= \begin{pmatrix} e^{i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\phi} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi} \\ 0 \\ e^{-i\phi} \end{pmatrix} \\ &= \hat{x}\cos\phi - \hat{y}\sin(\phi). \end{aligned}$$

5)

$$U = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -i/\sqrt{2} & 0 & i\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$$

6)

$$\begin{aligned} R'_z = UR_zU^\dagger &= \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -i/\sqrt{2} & 0 & i\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -i/\sqrt{2} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -i/\sqrt{2} & 0 & i\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\phi}/\sqrt{2} & ie^{i\phi}/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ e^{-i\phi}/\sqrt{2} & -ie^{-i\phi}/\sqrt{2} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}e^{i\phi} + \frac{1}{2}e^{-i\phi} & \frac{i}{2}e^{i\phi} - \frac{i}{2}e^{-i\phi} & 0 \\ \frac{-i}{2}e^{i\phi} + \frac{i}{2}e^{-i\phi} & \frac{1}{2}e^{i\phi} + \frac{1}{2}e^{-i\phi} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$