*QUANTUM MECHANICS SUBJECT EXAM*, August 25, 2020 9:00 AM - 12:00 PM

SECRET STUDENT NUMBER: STUDNUMBER

1. If you are taking the exam online, images of answers must be sent in within 15 minutes of the final finishing time by email to mailto:prattsc@msu.edu. The easiest way to submit answers is probably taking a picture by phone and sending photos in jpeg format or pdf format. Please use resolution of 300 dpi or file sizes of ≈ 500-600 kB per page. If you can make the images grayscale, that will help with file size. Solutions for different problems should appear as separate attachments. The MSU email system has a file size limit of 25 MB for one message. If possible, send all the images as part of a single email message. You should probably make sure your upload is using wifi, as cellular might be slow and unreliable.

2. **Do not write your name on the exam. Your exam has a "secret student number" on each page. Be sure to write that number on EACH additional page of the solutions you submit**.

3. This exam is closed-book, closed-notes and closed-internet. You are not permitted to use mathematical software, e.g. mathematica. You are not to communicate with any other individuals regarding the exam, during the exam.

$$\int_{-\infty}^{\infty} dx\; e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t,\; \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\; V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\; \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\; |x\rangle e^{ipx/\hbar},\quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2},\quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\begin{aligned}\vec{j}(\vec{r},t) &= \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) \\ &\quad - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.\end{aligned}$$

$$H = \frac{(\vec{P} - e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x-y):\quad -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A},\quad \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\text{cyclotron}} = \frac{eB}{mc},$$

$$e^{A+B} = e^A e^B e^{-C/2},\quad \text{if } [A,B] = C,\; \text{and } [C,A] = [C,B] = 0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}},\quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),\quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi},\quad Y_{\ell-m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi).$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t) = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\mathcal{V}(r)e^{i(\vec{k}_f-\vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k-\epsilon_r)^2+(\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{single}}\tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \left|\sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}}\right|^2,$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell+1)e^{i\delta_\ell}\sin\delta_\ell\frac{1}{k}P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r}f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,\quad \sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell,\quad \delta \approx -ak$$

$$L_\pm|\ell,m\rangle = \sqrt{\ell(\ell+1)-m(m\pm 1)}|\ell,m\pm 1\rangle,$$

$$C^{\ell,s}_{m_\ell,m_s;JM} = \langle \ell,s,J,M|\ell,s,m_\ell,m_s\rangle,$$

$$\langle\tilde{\beta},J,M|T^k_q|\beta,\ell,m_\ell\rangle = C^{k\ell}_{qm_\ell;JM}\frac{\langle\tilde{\beta},J||T^{(k)}||\beta,\ell,J\rangle}{\sqrt{2J+1}},$$

$$n = \frac{(2s+1)}{(2\pi)^d}\int_{k<k_f} d^dk,\quad d\text{ dimensions},$$

$$\{\Psi_s(\vec{x}),\Psi^\dagger_{s'}(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\delta_{ss'},$$

$$\Psi^\dagger_s(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}a^\dagger_s(\vec{k}),\quad \{\Psi_s(\vec{x}),a^\dagger_\alpha\} = \phi_{\alpha,s}(\vec{x}).$$

1. (10 pts) A particle of mass $m$ interacts with a spherically symmetric attractive potential,

$$V(r) = \begin{cases} -V_0 + \frac{V_0 r}{a}, & r < a \\ 0, & r > a \end{cases}$$

Circle the true statements below:

- Fixing $a$, and making $V_0$ very small, but non-zero, there will always be at least one bound state.
- Fixing $V_0$, and making $a$ very small, but non-zero, there will always be at least one bound state.
- Fixing $a$, as the magnitude of $V_0$ increases, the number of bound states will increase.
- Fixing $V_0$, as the magnitude of $a$ increases, the number of bound states will increase.

2. A system of $\boldsymbol{N}$ spinless particles are self-bound due to an attractive radially symmetric interaction. States are labeled $|\boldsymbol{\alpha}, \boldsymbol{L}, \boldsymbol{M}\rangle$, where $\boldsymbol{L}$ and $\boldsymbol{M}$ reference the total angular momentum and $\boldsymbol{\alpha}$ accounts for all other quantum labels.

   (a) (10 pts) For the matrix elements listed below, circle the NON-ZERO elements: ($\boldsymbol{J}, \boldsymbol{R}, \boldsymbol{P}$ are current, position and momentum operators)

   - $\langle \alpha', L' = 2, M' = 1 | P_x^2 + P_y^2 | \alpha, L = 4, M = 3\rangle$
   - $\langle \alpha', L' = 2, M' = 1 | P_x P_y | \alpha, L = 4, M = 1\rangle$
   - $\langle \alpha', L' = 2, M' = 1 | \epsilon_{ijk} J_i R_j P_k | \alpha, L = 2, M = 1\rangle$
   - $\langle \alpha', L' = 2, M' = 1 | P_x | \alpha, L = 3, M = 1\rangle$

   (b) (10 pts) With great effort, you calculated the matrix element

$$\mathcal{M} = \langle \alpha', L' = 2, M' = 0 | P_x^2 + P_y^2 - 2P_z^2 | \alpha, L = 4, M = 0\rangle$$

   by performing a long and difficult integral. If you were to use the Wigner Eckart theorem, circle the matrix elements below you could express in terms of $\mathcal{M}$ and Clebsch-Gordan coefficients without having to perform a new integral.

   - $\langle \alpha', L' = 2, M' = 2 | P_x^2 + P_y^2 - 2P_z^2 | \alpha, L = 4, M = 2\rangle$
   - $\langle \alpha', L' = 2, M' = 2 | P_x^2 + P_y^2 | \alpha, L = 4, M = 2\rangle$
   - $\langle \alpha', L' = 2, M' = 0 | P_x P_y | \alpha, L = 4, M = 2\rangle$
   - $\langle \alpha', L' = 2, M' = 0 | P_x^2 + P_y^2 - 2P_z^2 | \alpha, L = 2, M = 0\rangle$

3. (20 pts) A long array of point charges are positioned in a line along the $\boldsymbol{z}$ axis, with a distance $\boldsymbol{a}$ separating each charge. A beam with momentum $\boldsymbol{p}$ is incident on the charges along the $\boldsymbol{z}$ axis. The scattering can be considered as a perturbative process. In order to measure the distance $\boldsymbol{a}$, you measure the directions at which the differential cross section is the largest. In terms of $\boldsymbol{a}$ and $\boldsymbol{p}$, list the angles for which all the scattering is strongest, i.e. where all the charges contribute coherently.

$$q_z = p(1 - \cos\theta)/\hbar$$

$$q_z a = 2n\pi$$

$$1 - \cos\theta_n = \frac{\hbar \cdot 2n\pi}{pa}$$

$$\theta_n = \cos^{-1}\left\{1 - \frac{2n\pi\hbar}{pa}\right\}$$

**Extra space for #3**

4. Consider a TWO-DIMENSIONAL world with protons and neutrons of mass $\boldsymbol{M}$ and massless electrons and neutrinos. Neutrinos and anti-neutrinos can readily exit or enter the system. The system is confined to a large box of volume $\boldsymbol{\Omega}$, has zero net electric charge, and has a baryon density (number of baryons per area) of $\boldsymbol{\rho_B}$. The protons and neutrons move non-relativistically. The interactions,

$$\boldsymbol{p + e \leftrightarrow n + \nu, \quad n \leftrightarrow p + e + \bar{\nu},}$$

take place until the energy is minimized.

For each question below, give your answer in terms of $\boldsymbol{\rho_B}$, $\boldsymbol{M}$, $\hbar$, $\boldsymbol{\Omega}$, and the Fermi wave numbers $\boldsymbol{k_p}$, $\boldsymbol{k_e}$ and $\boldsymbol{k_f}$ for protons, electrons and neutrons respectively.

(a) (15 pts) Write three equations expressing the fact that the system is electrically neutral, has fixed net areal baryon density $\boldsymbol{\rho_B}$, and has minimum energy.

(b) (5 pts) Solve for $\boldsymbol{k_p, k_n}$ and $\boldsymbol{k_e}$.

a) $k_e = k_p$

$$\rho_B = \frac{2}{(2\pi)^2}\left[\pi k_n^2 + \pi k_p^2\right]$$

$$= \frac{1}{2\pi}\left(k_p^2 + k_n^2\right)$$

$$\frac{\hbar^2}{2M} k_n^2 = \hbar k_e c + \frac{\hbar^2 k_p^2}{2M}$$

b) $$2\pi\rho_B = k_p^2 + k_n^2$$

$$-\frac{2M}{\hbar^2}\hbar c k_p = k_p^2 - k_n^2$$

$$k_p^2 + \frac{Mc}{\hbar} k_p - \pi\rho_B = 0$$

$$k_p = \frac{1}{2}\left[-\frac{Mc}{\hbar} + \sqrt{\left(\frac{Mc}{\hbar}\right)^2 + 4\pi\rho_B\frac{Mc}{\hbar}}\right]$$

$$k_e = k_p$$

$$2k_n^2 = 2\pi\rho_B + \frac{2Mc}{\hbar}k_p$$

$$k_n = \sqrt{\pi\rho_B + \frac{Mc}{\hbar}k_p}$$

**Extra space for #4**

5. (30 pts) Consider a ONE-DIMENSIONAL world, where a non-relativistic particle of mass $M$ is in the ground state of a harmonic oscillator characterized by frequency $\omega_0$. The harmonic oscillator is in large box of length $L$ which is populated by a bath of massless particles. The probability that any given state in the box is occupied is $f(k)$, where $k$ is the wave number of the massless particle. The harmonic oscillator can be excited to the first excited state via the weak coupling,

$$V = g\int dx\, \Psi^\dagger(x)x\Psi(x)\Phi(x),$$

where $\Psi$ is the field operator for the massive particle and $[\Psi(x,t),\Psi^\dagger(x',t)] = \delta(x-x')$, and $\Phi$ is the field operator for the massless particle,

$$\Phi(x,t) = \sum_k \frac{1}{\sqrt{2E_kL}}\left[a(\cancel{k}_k)e^{-i\omega_t+ikx} + a^\dagger(\cancel{k}_k)e^{i\omega_t-ikx}\right],$$
$$[a(\cancel{k}_k), a^\dagger(\cancel{k'}_{k'})] = \delta_{kk'}.$$

In the dipole approximation

Using Fermi's golden rule find the rate at which the massive particle is excited to the first excited state from the ground state. Your answer should be in terms of $m$, $\omega_0$, $g$ and $f(k)$.

$$\mathcal{M} = \langle n=1|V|n=0,k\rangle$$
$$= g\int dx\, \cancel{e^{ikx}}\, x\, \psi_1^*(x)\psi(x)\frac{1}{\sqrt{2E_kL}}$$
dipole approx
$$= \frac{g}{\sqrt{2E_kL}}\cdot\langle 1|X|0\rangle, \quad X = \sqrt{\frac{\hbar}{2M\omega_0}}(a+a^\dagger)$$
$$= \frac{g}{\sqrt{2E_kL}}\sqrt{\frac{\hbar}{2M\omega}}$$
$$|\mathcal{M}|^2 = \frac{\hbar g^2}{4ME_kL\omega}$$
$$\Gamma = \frac{2\pi}{\hbar}\sum_k f(k)\frac{\hbar g^2}{4ME_kL\omega_0}\delta(E_k-\hbar\omega_0)$$
$$= \frac{2\pi g^2}{4ME_k\omega_0}\int_0^\infty \frac{2dk}{2\pi}\delta(\hbar ck-\hbar\omega_0).$$
$$= \frac{g^2}{2M\hbar^2c^2k\omega_0}f(k), \quad k=\omega_0/c.$$
$$= \boxed{g^2 f(\omega_0/c)\big/2M\hbar^2\omega_0^2 c}$$

**Extra space for #5**