MIDTERM EXAM,

PHYSICS 852, Spring 2022

Friday, March 4, 11:30-12:20 PM

Your Name: ______________________________

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t,\ \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\ V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\ \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\ |x\rangle e^{ipx/\hbar},\quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2},\quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.$$

$$H = \frac{(\vec{P} - e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x-y):\quad -\frac{\hbar^2}{2m}\left(\partial_x\psi(x)|_{y+\epsilon} - \partial_x\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A},\quad \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\rm cyclotron} = \frac{eB}{mc},$$

$$e^{A+B} = e^Ae^Be^{-C/2},\quad \text{if } [A,B] = C, \text{ and } [C,A] = [C,B] = 0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}},\quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),\quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm2i\phi},\quad Y_{\ell -m}(\theta,\phi) = (-1)^mY^*_{\ell m}(\theta,\phi),$$

$$|N\rangle = |n\rangle - \sum_{m\neq n}|m\rangle\frac{1}{\epsilon_m - \epsilon_n}\langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n}\frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x)=\frac{\sin x}{x},\ n_0(x)=-\frac{\cos x}{x},\ j_1(x)=\frac{\sin x}{x^2}-\frac{\cos x}{x},\ n_1(x)=-\frac{\cos x}{x^2}-\frac{\sin x}{x}$$

$$j_2(x)=\left(\frac{3}{x^3}-\frac{1}{x}\right)\sin x-\frac{3}{x^2}\cos x,\ n_2(x)=-\left(\frac{3}{x^3}-\frac{1}{x}\right)\cos x-\frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t)=\frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n-E_i),$$

$$\frac{d\sigma}{d\Omega}=\frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\mathcal{V}(r)e^{i(\vec{k}_f-\vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma=\frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k-\epsilon_r)^2+(\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega}=\left(\frac{d\sigma}{d\Omega}\right)_{\rm single}\tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q})=\frac{1}{N}\left|\sum_{\vec{a}}e^{i\vec{q}\cdot\vec{a}}\right|^2=\sum_{\delta\vec{a}}e^{i\vec{q}\cdot\delta\vec{a}},$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm Rutherford}=\frac{e^4Z_1^2Z_2^2m^2}{(\hbar k)^4(1-\cos\theta)^2}$$

$$\frac{d\sigma}{d\Omega}=\left(\frac{d\sigma}{d\Omega}\right)_{\rm Rutherford}\left|\frac{1}{e}\int d^3r\ \rho(\vec{r})e^{i\vec{q}\cdot\vec{r}}\right|^2$$

$$e^{i\vec{k}\cdot\vec{r}}=\sum_\ell(2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta)=\sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x)=1,\ P_1(x)=x,\ P_2(x)=(3x^2-1)/3,$$

$$f(\Omega)\equiv\sum_\ell(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell\frac{1}{k}P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty}=e^{i\vec{k}\cdot\vec{r}}+\frac{e^{ikr}}{r}f(\Omega),$$

$$\frac{d\sigma}{d\Omega}=|f(\Omega)|^2,\quad \sigma=\frac{4\pi}{k^2}\sum_\ell(2\ell+1)\sin^2\delta_\ell,\quad \delta\approx-ak$$

$$L_\pm|\ell,m\rangle=\sqrt{\ell(\ell+1)-m(m\pm1)}|\ell,m\pm1\rangle,$$

$$\langle\tilde{\beta},J,M|T_q^k|\beta,\ell,m_\ell\rangle=\langle JM|k,q,\ell,m_\ell\rangle\frac{\langle\tilde{\beta},J||T^{(k)}||\beta,\ell,J\rangle}{\sqrt{2J+1}},$$

$$n=\frac{(2s+1)}{(2\pi)^d}\int_{k<k_f}d^dk,\quad d\ \text{dimensions},$$

$$\{\Psi_s(\vec{x}),\Psi_{s'}^\dagger(\vec{y})\}=\delta^3(\vec{x}-\vec{y})\delta_{ss'},$$

$$\Psi_s^\dagger(\vec{r})=\frac{1}{\sqrt{V}}\sum_{\vec{k}}e^{i\vec{k}\cdot\vec{r}}a_s^\dagger(\vec{k}),\quad \{\Psi_s(\vec{x}),a_\alpha^\dagger\}=\phi_{\alpha,s}(\vec{x}).$$

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1. Type $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ particles exist in a ONE-DIMENSIONAL world. The $\boldsymbol{\alpha}$ particle has mass $\boldsymbol{M_\alpha}$ and is described by the one-dimensional field operator (in the interaction representation) within a large length $\boldsymbol{L}$,

$$\begin{aligned} \Phi_\alpha(x,t) &= \frac{1}{\sqrt{L}}\sum_k e^{-iE_k t/\hbar + ikx} a_k, \\ \Phi_\alpha^\dagger(x,t) &= \frac{1}{\sqrt{L}}\sum_k e^{iE_k t/\hbar - ikx} a_k^\dagger, \\ E_k &= M_\alpha c^2 + \frac{\hbar^2 k^2}{2M_\alpha}. \end{aligned}$$

The $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ particles are massless and described by the operators,

$$\begin{aligned} \Psi_\beta(x,t) &= \frac{1}{\sqrt{L}}\sum_q e^{-iE_q t/\hbar + iqx} b_q, \\ \Psi_\beta^\dagger(x,t) &= \frac{1}{\sqrt{L}}\sum_q e^{iE_q t/\hbar - iqx} b_q^\dagger, \\ \Psi_\gamma(x,t) &= \frac{1}{\sqrt{L}}\sum_q e^{-iE_q t/\hbar + iqx} c_q, \\ \Psi_\gamma^\dagger(x,t) &= \frac{1}{\sqrt{L}}\sum_q e^{iE_q t/\hbar - iqx} c_q^\dagger, \\ E_q &= \hbar c q. \end{aligned}$$

The massive $\boldsymbol{\alpha}$ particle can decay to a $\boldsymbol{\beta}$ and a $\boldsymbol{\gamma}$ particle via the interaction

$$H_{\text{int}} = g\int dx\ \left[\Phi_\alpha(x,t)\Psi_\beta^\dagger(x,t)\Psi_\gamma^\dagger(x,t) + \Phi_\alpha^\dagger(x,t)\Psi_\beta(x,t)\Psi_\gamma(x,t)\right],$$

where the coupling constant $\boldsymbol{g}$ is small. The creation and destruction operators obey the commutation rules $[a_k, a_{k'}^\dagger] = \delta_{kk'}$, $[b_q, b_{q'}^\dagger] = \delta_{qq'}$ and $[c_q, c_{q'}^\dagger] = \delta_{qq'}$.

(a) (5 pts) Evaluate the commutator $[\Phi_\alpha(x,t), \Phi_\alpha^\dagger(x',t)]$.

(b) (5 pts) What is the dimensionality of $\boldsymbol{g}$?

(c) (25 pts) Calculate the rate at which an $\boldsymbol{\alpha}$ particle at rest decays into a $\boldsymbol{\beta}$ and a $\boldsymbol{\gamma}$ particle.

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Solution:

a)

$$
\begin{aligned}
[\Phi_\alpha(x,t), \Phi^\dagger_\alpha(x',t)] &= \frac{1}{L}\sum_{kk'} e^{ikx-ik'x'-i\omega_k t+i\omega_{k'}t}[a_k, a^\dagger_{k'}] \\
&= \frac{1}{L}\sum_k e^{ik(x-x')} \\
&= \frac{1}{2\pi}\int dk\; e^{ik(x-x')} \\
&= \delta(x-x').
\end{aligned}
$$

b)

$$
\begin{aligned}
[E] &= [g][L][1/L^{3/2}], \\
[g] &= [E][L]^{1/2}.
\end{aligned}
$$

c) First calculate matrix element

$$
\begin{aligned}
&\langle \beta, q; \gamma, q'|H_{\text{int}}|\alpha, P=0\rangle \\
&\qquad = \frac{g}{L^{3/2}}\int dx\; \langle 0|b_q c_{q'}\left(\sum_{kk'K} e^{-i(k+k'-K)x} b^\dagger_k c^\dagger_{k'} a_K\right) a^\dagger_{P=0}|0\rangle \\
&\qquad = \frac{g}{L^{3/2}}\int dx\; e^{-i(q+q')x} \\
&\qquad = \frac{g}{L^{1/2}}\delta_{q,-q'}.
\end{aligned}
$$

Now, apply Fermi's golden rule.

$$
\begin{aligned}
\Gamma &= \frac{2\pi}{\hbar}\frac{g^2}{L}\sum_{qq'}\delta_{q,-q'}\delta(Mc^2 - E_q - E_{q'}) \\
&= \frac{2\pi}{\hbar}\frac{g^2}{L}\sum_{q}\delta(Mc^2 - 2E_q), \\
&= \frac{g^2}{\hbar}\int_{-\infty}^{\infty} dq\; \delta(Mc^2 - 2\hbar c q) \\
&= \frac{2g^2}{\hbar}\frac{1}{2\hbar c} \\
&= \frac{g^2}{\hbar^2 c}.
\end{aligned}
$$

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2A (20 pts) Imagine you had calculated the following matrix element,

$$\mathcal{M} = \langle \alpha, J_f = 1, M_f = 0 | x^2 + y^2 - 2z^2 | \beta, J_i = 3, M_i = 0 \rangle.$$

For the following matrix elements, first state whether they are zero, and if not, express them in terms of $\mathcal{M}$ and Clebsch-Gordan coefficients. **You do NOT need to evaluate any Clebsch-Gordan coefficients in your answers.**

- $\langle \alpha, J_f = 1, M_f = 0 | x^2 + y^2 + z^2 | \beta, J_i = 3, M_i = 0 \rangle$

- $\langle \alpha, J_f = 1, M_f = 0 | z^2 | \beta, J_i = 3, M_i = 0 \rangle$

- $\langle \alpha, J_f = 1, M_f = 1 | xz | \beta, J_i = 3, M_i = 1 \rangle$

- $\langle \alpha, J_f = 1, M_f = 1 | xy | \beta, J_i = 3, M_i = 2 \rangle$

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Solution:

$$0 = \langle \alpha, J_f = 1, M_f = 0 | x^2 + y^2 + z^2 | \beta, J_i = 3, M_i = 0 \rangle$$

because the operator is a scalar.

$$\begin{aligned} 0 &= (x^2 + y^2 + z^2) = (x^2 + y^2 - 2z^2 + 3z^2) \\ z^2 &= (x^2 + y^2 + z^2)/3 - (x^2 + y^2 - 2z^2)/3, \\ \langle \alpha, J_f &= 1, M_f = 0 | z^2 | \beta, J_i = 3, M_i = 0 \rangle \\ &= -\langle \alpha, J_f = 1, M_f = 0 | x^2 + y^2 - 2z^2 | \beta, J_i = 3, M_i = 0 \rangle / 3 \\ &= -\mathcal{M}/3. \end{aligned}$$

The last two matrix elements are zero.

$$\langle \alpha, J_f = 1, M_f = 1 | xz | \beta, J_i = 3, M_i = 1 \rangle = 0$$

because xz is comprised of $T^2_{\pm 1}$

$$\langle \alpha, J_f = 1, M_f = 1 | xy | \beta, J_i = 3, M_i = 2 \rangle$$

because xy is comprised of $T^2_{\pm 2}$

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2B (15 pts) Now, imagine you had calculated a matrix element,

$$\boldsymbol{\mathcal{V} = \langle \alpha, J_f = 1, M_f = 0 | P_z | \beta, J_i = 0, M_i = 0 \rangle.}$$

For all three values of $\boldsymbol{M_f}$, express the following matrix elements in terms of $\boldsymbol{\mathcal{V}}$ and Clebsch-Gordan coefficients. **Again, You do NOT need to evaluate any Clebsch-Gordan coefficients in your answers.**
$\boldsymbol{\langle \alpha, J_f = 1, M_f | P_x | \beta, J_i = 0, M_i = 0 \rangle}$. Be sure to note when a matrix element vanishes.

Solution:

$$\begin{aligned}
P_z &= |P|\sqrt{\frac{4\pi}{3}} Y_{\ell=1,m=0}, \\
\sin\theta e^{\pm i\phi} &= \sqrt{\frac{8\pi}{3}} \mp Y_{1,\pm 1}, \\
P_x &= |P|\sqrt{\frac{2\pi}{3}} (-Y_{\ell=1,m=1} + Y_{\ell=1,m=-1}).
\end{aligned}$$

The three elements are now:

$$\begin{aligned}
\langle J_f = 1, M_f = 1 | P_x | J_i = 0, M_i = 0 \rangle &= \frac{-\mathcal{V}}{\sqrt{2}} \frac{\langle 1, 1 | 1, 1, 0, 0 \rangle}{\langle 1, 0 | 1, 0, 0, 0 \rangle}, \\
\langle J_f = 1, M_f = -1 | P_x | J_i = 0, M_i = 0 \rangle &= \frac{\mathcal{V}}{\sqrt{2}} \frac{\langle 1, -1 | 1, -11, 0, 0 \rangle}{\langle 1, 0 | 1, 0, 0, 0 \rangle}, \\
\langle J_f = 1, M_f = 0 | P_x | J_i = 0, M_i = 0 \rangle &= 0.
\end{aligned}$$

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3 In the center of a ONE-DIMENSIONAL star, matter consists of weakly interacting electrons, protons and neutrons. The baryon density is n_B baryons per length. Interactions of the type

$$n \rightarrow e + p + \bar{\nu}, \quad e + p \rightarrow n + \nu,$$

can proceed with the neutrinos leaving the star until the energy is minimized for the given n_B. Assume the neutrons and protons have the same mass M, and can be treated non-relativistically, whereas the electrons can be treated as if they are massless. Refer to the three Fermi wave numbers as k_e, k_n and k_p.

(a) (15 pts) Write three equations involving k_e, k_n and k_p, which when solved can yield the three Fermi wave numbers.

(b) (15 pts) Solve for the neutron fraction, i.e. the fraction of baryons that are neutrons.

Solution:
a) 1. Charge conservation: $k_e = k_p$
One-dimensional density,

$$n = \frac{2}{\pi}\int_0^{k_f} dk = \frac{2}{\pi}k_f.$$

2. Baryon conservation $n_B = \frac{2}{\pi}(k_n + k_p)$
3 Minimize energy

$$\frac{\hbar^2 k_n^2}{2M} = \frac{\hbar^2 k_p^2}{2M} + \hbar c k_e.$$

b)

$$\frac{\hbar^2}{2M}\left[\left(\frac{\pi n_B}{2} - k_p\right)^2 - k_p^2\right] - \hbar c k_p = 0,$$

$$\frac{\hbar^2}{2M}\left[-\pi n_B k_p + \left(\frac{\pi n_B}{2}\right)^2\right] - \hbar c k_p = 0,$$

$$k_p\left(\frac{\hbar^2}{2M}\pi n_B + \hbar c\right) = \frac{\hbar^2}{2M}\left(\frac{\pi n_B}{2}\right)^2,$$

$$
\begin{aligned}
\boldsymbol{k_p} &= \frac{\frac{\hbar^2}{2M}\left(\frac{\pi n_B}{2}\right)^2}{\left(\frac{\hbar^2}{2M}\pi n_B + \hbar c\right)} \\
&= \frac{\pi n_B/2}{2 + 4Mc/(\pi\hbar n_B)}, \\
\boldsymbol{k_n} &= \frac{\pi}{2} n_B - k_p \\
&= \frac{\pi n_B}{2}\left(1 - \frac{1}{2 + 4Mc/(\pi\hbar n_B)}\right) \\
&= \frac{\pi n_B}{2}\frac{1 + 4Mc/(\pi\hbar n_B)}{2 + 4Mc/(\pi\hbar n_B)} \\
\frac{n_n}{n_B} &= \frac{1 + 4Mc/(\pi\hbar n_B)}{2 + 4Mc/(\pi\hbar n_B)}.
\end{aligned}
$$