MIDTERM EXAM,
PHYSICS 852, Spring 2021
Friday, March 5, 1:50-2:40 PM

SECRET STUDENT NUMBER:
STUDNUMBER

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t,\ \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\ V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\ \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\ |x\rangle e^{ipx/\hbar},\quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2},\quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.$$

$$H = \frac{(\vec{P}-e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x-y):\quad -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A},\quad \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\text{cyclotron}} = \frac{eB}{mc},$$

$$e^{A+B} = e^A e^B e^{-C/2},\quad \text{if } [A,B] = C,\ \text{and } [C,A] = [C,B] = 0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}},\quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{1,\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),\quad Y_{2,\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm 2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi},\quad Y_{\ell -m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi).$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t) = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\mathcal{V}(r)e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{single}}\tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \left|\sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}}\right|^2,$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell+1)e^{i\delta_\ell}\sin\delta_\ell\frac{1}{k}P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r}f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,\quad \sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell,\quad \delta \approx -ak$$

$$L_\pm|\ell,m\rangle = \sqrt{\ell(\ell+1) - m(m\pm1)}|\ell,m\pm1\rangle,$$

$$C^{\ell,s}_{m_\ell,m_s;JM} = \langle \ell,s,J,M|\ell,s,m_\ell,m_s\rangle,$$

$$\langle\tilde{\beta},J,M|T^k_q|\beta,\ell,m_\ell\rangle = C^{k\ell}_{qm_\ell;JM}\frac{\langle\tilde{\beta},J||T^{(k)}||\beta,\ell,J\rangle}{\sqrt{2J+1}},$$

$$n = \frac{(2s+1)}{(2\pi)^d}\int_{k<k_f} d^dk,\quad d \text{ dimensions},$$

$$\{\Psi_s(\vec{x}),\Psi^\dagger_{s'}(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\delta_{ss'},$$

$$\Psi^\dagger_s(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}a^\dagger_s(\vec{k}),\quad \{\Psi_s(\vec{x}),a^\dagger_\alpha\} = \phi_{\alpha,s}(\vec{x}).$$

1. Type α, β and γ particles exist in a TWO-DIMENSIONAL world. The α particle has mass M_α and is described by the two-dimensional field operator within a large area A,

$$\Phi_\alpha(\vec{r},t) = \frac{1}{\sqrt{A}}\sum_{\vec{k}} e^{-iE_k t/\hbar + i\vec{k}\cdot\vec{r}} a_{\vec{k}},$$
$$\Phi_\alpha^\dagger(\vec{r},t) = \frac{1}{\sqrt{A}}\sum_{\vec{k}} e^{iE_k t/\hbar - i\vec{k}\cdot\vec{r}} a_{\vec{k}}^\dagger,$$
$$E_k = M_\alpha c^2 + \frac{\hbar^2 k^2}{2M_\alpha}.$$

The β and γ particles are massless and described by the operators,

$$\Psi_\beta(\vec{r},t) = \frac{1}{\sqrt{A}}\sum_{\vec{q}} e^{-iE_q t/\hbar + i\vec{q}\cdot\vec{r}} b_q,$$
$$\Psi_\beta^\dagger(\vec{r},t) = \frac{1}{\sqrt{A}}\sum_{\vec{q}} e^{iE_q t/\hbar - i\vec{q}\cdot\vec{r}} b_q^\dagger,$$
$$\Psi_\gamma(\vec{r},t) = \frac{1}{\sqrt{A}}\sum_{\vec{q}} e^{-iE_q t/\hbar + i\vec{q}\cdot\vec{r}} c_q,$$
$$\Psi_\gamma^\dagger(\vec{r},t) = \frac{1}{\sqrt{A}}\sum_{\vec{q}} e^{iE_q t/\hbar - i\vec{q}\cdot\vec{r}} c_q^\dagger,$$
$$E_q = \hbar c q.$$

The massive α particle can decay to an β and a γ particle via the interaction

$$H_{\rm int} = g\int dxdy\,\left[\Phi_\alpha(\vec{r},t)\Psi_\beta^\dagger(\vec{r},t)\Psi_\gamma^\dagger(\vec{r},t) + \Phi_\alpha^\dagger(\vec{r},t)\Psi_\beta(\vec{r},t)\Psi_\gamma(\vec{r},t)\right],$$

where the coupling constant g is small. The creation and destruction operators obey the commutation rules $[a_{\vec{k}}, a_{\vec{k}'}^\dagger] = \delta_{\vec{k}\vec{k}'}$, $[b_{\vec{q}}, b_{\vec{q}'}^\dagger] = \delta_{\vec{q}\vec{q}'}$ and $[c_{\vec{q}}, c_{\vec{q}'}^\dagger] = \delta_{\vec{q}\vec{q}'}$. FYI: In two dimensions $\delta_{\vec{k}\vec{k}'} = \delta_{k_x k_x'}\delta_{k_y k_y'}$.

(a) (5 pts) Evaluate the commutator $[\Phi_\alpha(\vec{r},t), \Phi_\alpha^\dagger(\vec{r}',t)]$.

(b) (5 pts) What is the dimensionality of g?

(c) (20 pts) Calculate the rate at which an α particle at rest decays into a β and a γ particle.

(d) (5 pts) How would your answer change if the β and γ particles were identical?

(Extra work space for #1)

(Extra work space for #1)

2. Imagine you had calculated the following matrix element,

$$\boldsymbol{\mathcal{M} = \langle \alpha, J_f = 1, M_f = 0 | z^2 | \beta, J_i = 3, M_i = 0 \rangle.}$$

For the following matrix elements, first state whether they are zero, and if not, express them in terms of $\boldsymbol{\mathcal{M}}$. **Evaluate any Clebsch-Gordan coefficients in your answers. You can use tables or online resources to get the values.** (5 pts. each)

- $\langle \alpha, J_f = 1, M_f = 0 | x^2 + y^2 | \beta, J_i = 3, M_i = 0 \rangle$

- $\langle \alpha, J_f = 1, M_f = 0 | z^2 | \beta, J_i = 3, M_i = 2 \rangle$

- $\langle \alpha, J_f = 1, M_f = 0 | xz | \beta, J_i = 3, M_i = 0 \rangle$

- $\langle \alpha, J_f = 1, M_f = 0 | xz | \beta, J_i = 3, M_i = 2 \rangle$

- $\langle \alpha, J_f = 1, M_f = 0 | xy | \beta, J_i = 3, M_i = 2 \rangle$

(Extra work space for #2)

3. Electrons of mass $\boldsymbol{m}$ are placed in a three-dimensional spherically symmetric harmonic oscillator with potential, $\boldsymbol{V = m\omega^2 r^2/2}$.

   (a) (5 pts) What is the ground state energy for a single electron?

   (b) (5 pts) What is the ground state energy if 20 electrons are in the well? (sum over all particles, $\boldsymbol{\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{20}}$)

   Now add an interaction with an external magnetic field,

$$\boldsymbol{V_\beta = -\mu \vec{B} \cdot (\vec{L} + 2\vec{S})},$$

   where the strength of the magnetic field is adjusted so that $\boldsymbol{\mu B = \omega}$.

   (c) (10 pts) What is the new single-particle ground state energy?

   (d) (10 pts) What is the ground state energy if 20 electrons are in the well?

(Extra work space for #3)