

PHY 851 – QUANTUM MECHANICS
MIDTERM II

October 23-25, 2020

- Submit your exam via GRADESCOPE. Be sure that no page contains responses for more than one problem. Having different parts 2b,2c... of a given problem on a single page is fine. Individual problems can span multiple pages.
- Once the exam is opened, you are to upload your answer within six hours.
- **Do not write your name on the exam**
- This exam is open-book, open-notes, open-internet but closed-mouth. You are permitted to use mathematical software, e.g. Mathematica. You are not to communicate with any other individuals regarding the exam, during the exam.

$$\int_{-\infty}^{\infty} dx e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t, \quad \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t, -\infty) = 1 + \frac{-i}{\hbar} \int_{-\infty}^t dt' V(t')U(t', -\infty),$$

$$\langle x|x'\rangle = \delta(x - x'), \quad \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p - p'),$$

$$|p\rangle = \int dx |x\rangle e^{ipx/\hbar}, \quad |x\rangle = \int \frac{dp}{2\pi\hbar} |p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} X - i\sqrt{\frac{1}{2\hbar m\omega}} P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}} e^{-x^2/2b^2}, \quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r}, t) = \psi^*(\vec{r}_1, t_1)\psi(\vec{r}_2, t_2)$$

$$\vec{j}(\vec{r}, t) = \frac{-i\hbar}{2m} (\psi^*(\vec{r}, t)\nabla\psi(\vec{r}, t) - (\nabla\psi^*(\vec{r}, t))\psi(\vec{r}, t))$$

$$- \frac{e\vec{A}}{mc} |\psi(\vec{r}, t)|^2.$$

$$H = \frac{(\vec{P} - e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x - y) : \quad -\frac{\hbar^2}{2m} \left(\frac{\partial}{\partial x} \psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x} \psi(x)|_{y-\epsilon} \right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A}, \quad \vec{B} = \nabla \times \vec{A},$$

$$\omega_{\text{cyclotron}} = \frac{eB}{mc},$$

$$e^{A+B} = e^A e^B e^{-C/2}, \quad \text{if } [A, B] = C, \text{ and } [C, A] = [C, B] = 0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}, \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos\theta, \quad Y_{1,\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1), \quad Y_{2,\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{\pm 2i\phi}, \quad Y_{\ell-m}(\theta, \phi) = (-1)^m Y_{\ell m}^*(\theta, \phi).$$

$$|N\rangle = |n\rangle - \sum_{m \neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \dots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m \neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x}, \quad n_0(x) = -\frac{\cos x}{x}, \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right) \sin x - \frac{3}{x^2} \cos x, \quad n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right) \cos x - \frac{3}{x^2} \sin x,$$

$$\frac{d}{dt} P_{i \rightarrow n}(t) = \frac{2\pi}{\hbar} |V_{ni}|^2 \delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2 \hbar^4} \left| \int d^3r \mathcal{V}(r) e^{i(\vec{k}_f - \vec{k}_i) \cdot \vec{r}} \right|^2,$$

$$\sigma = \frac{(2S_R + 1) 4\pi (\hbar\Gamma_R/2)^2}{(2S_1 + 1)(2S_2 + 1) k^2 (\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega} \right)_{\text{single}} \tilde{S}(\vec{q}), \quad \tilde{S}(\vec{q}) = \left| \sum_{\vec{a}} e^{i\vec{q} \cdot \vec{a}} \right|^2,$$

$$e^{i\vec{k} \cdot \vec{r}} = \sum_{\ell} (2\ell + 1) i^{\ell} j_{\ell}(kr) P_{\ell}(\cos \theta),$$

$$P_{\ell}(\cos \theta) = \sqrt{\frac{4\pi}{2\ell + 1}} Y_{\ell, m=0}(\theta, \phi),$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = (3x^2 - 1)/3,$$

$$f(\Omega) \equiv \sum_{\ell} (2\ell + 1) e^{i\delta_{\ell}} \sin \delta_{\ell} \frac{1}{k} P_{\ell}(\cos \theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R \rightarrow \infty} = e^{i\vec{k} \cdot \vec{r}} + \frac{e^{ikr}}{r} f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2, \quad \sigma = \frac{4\pi}{k^2} \sum_{\ell} (2\ell + 1) \sin^2 \delta_{\ell}, \quad \delta \approx -ak$$

$$L_{\pm} |\ell, m\rangle = \sqrt{\ell(\ell + 1) - m(m \pm 1)} |\ell, m \pm 1\rangle,$$

$$C_{m_{\ell}, m_s; JM}^{\ell, s} = \langle \ell, s, J, M | \ell, s, m_{\ell}, m_s \rangle,$$

$$\langle \tilde{\beta}, J, M | T_q^k | \beta, \ell, m_{\ell} \rangle = C_{qm_{\ell}; JM}^{k\ell} \frac{\langle \tilde{\beta}, J || T^{(k)} || \beta, \ell, J \rangle}{\sqrt{2J + 1}},$$

$$n = \frac{(2s + 1)}{(2\pi)^d} \int_{k < k_f} d^d k, \quad d \text{ dimensions,}$$

$$\{\Psi_s(\vec{x}), \Psi_{s'}^{\dagger}(\vec{y})\} = \delta^3(\vec{x} - \vec{y}) \delta_{ss'},$$

$$\Psi_s^{\dagger}(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} e^{i\vec{k} \cdot \vec{r}} a_s^{\dagger}(\vec{k}), \quad \{\Psi_s(\vec{x}), a_{\alpha}^{\dagger}\} = \phi_{\alpha, s}(\vec{x}).$$

1. A particle of mass m and charge e feels a strong constant electric field $\vec{E} = E_0 \hat{x}$. Additionally, the particle experiences a constant weak magnetic field $\vec{B} = B_0 \hat{z}$. The magnetic field strength is less than the electric field strength, $E_0 > B_0$.

(a) (10 pts) Describe a reference frame where either \vec{E} or \vec{B} is zero. State the velocity (both magnitude and direction) of that frame, and describe the fields in that frame.

(b) Averaged over a sufficiently long time interval Δ to avoid any oscillatory terms, answer these questions about the components of the average velocity ($\bar{v}_x(t)$, $\bar{v}_y(t)$, $\bar{v}_z(t)$) at large times, $t \gg \Delta t$. Assume the particle might have some original momentum with non-zero components in each direction. Circle the correct answers:

i. (5 pts) Which components average to zero? [v_x , v_y , v_z]

ii. (5 pts) Which components average to a constant? (but not necessarily zero) [v_x , v_y , v_z]

iii. (5 pts) Which components continue to accelerate over time? [v_x , v_y , v_z]

You might choose a gauge where the vector potential \vec{A} is pointed along the y axis.

y

2. Consider a particle of mass m_0 confined to a two-dimensional harmonic oscillator,

$$V(x, y) = \frac{1}{2}m\omega^2(r^2), \quad r = \sqrt{x^2 + y^2}.$$

Consider solutions of the form,

$$\psi_{nm}(x, y) = \phi_{nm}(\mathbf{r})e^{im\phi}, \quad m = \text{an integer.}$$

- (a) (10 pts) Write a one-dimensional differential equation (Schrödinger's equation) in terms of \mathbf{r} for $\phi_{nm}(\mathbf{r})$ assuming ϕ_{nm} is an eigenstate with energy E_{nm} . Here, \mathbf{n} denotes different eigenstates with the same m .
- (b) (10 pts) What are the eigen-energies? Also state the degeneracies of each level. (easier to work out in Cartesian basis)
- (c) (5 pts) Consider the state (un-normalized)

$$\begin{aligned} \psi_{NM} &= (\mathbf{a}_x^\dagger + i\mathbf{a}_y^\dagger)^M (\mathbf{a}_x^\dagger \mathbf{a}_x^\dagger + \mathbf{a}_y^\dagger \mathbf{a}_y^\dagger)^{(N-M)/2} |0\rangle, \quad M \geq 0 \\ \psi_{NM} &= (\mathbf{a}_x^\dagger - i\mathbf{a}_y^\dagger)^{|M|} (\mathbf{a}_x^\dagger \mathbf{a}_x^\dagger + \mathbf{a}_y^\dagger \mathbf{a}_y^\dagger)^{(N-M)/2} |0\rangle, \quad M < 0 \end{aligned}$$

with energy $(N + 1)\hbar\omega$. Here, \mathbf{a}_x^\dagger and \mathbf{a}_y^\dagger are the raising operators in the Cartesian basis. Show that under rotations that ψ_{NM} behaves as $e^{iM\phi}$. I.e., show that

$$\mathbf{R}(\Delta\phi)\psi_{NM}(r, \phi) = e^{iM\Delta\phi}\psi_{NM}(r, \phi).$$

You may use the fact that \mathbf{a}_x^\dagger and \mathbf{a}_y^\dagger rotate amongst each other like \mathbf{x} and \mathbf{y} .

- (d) (5 pts) For a given eigen-energy $(N + 1)\hbar\omega$, what values of M are possible?

3. (20 pts) Two distinguishable spinless particles of mass m occupy the same $\ell = 1$ eigenstate of a spherically symmetric potential. An spin-spin interaction is then applied,

$$\mathbf{V}_{ss} = -\beta \vec{\mathbf{L}}_1 \cdot \vec{\mathbf{L}}_2.$$

- (a) (5 pts) What values of the total angular momentum might the system attain?
- (b) (15 pts) What are the shifts of the energy due to \mathbf{V}_{ss} ? Assume you needn't worry about mixing with other levels.

4. (20 pts) Two spinless particles (“1” and “2”) of mass m occupy an $\ell = 1$ eigenstate and an $\ell = 2$ eigenstate of a spherically symmetric potential. A magnetic field with the following interaction is then applied,

$$\mathbf{V}_b = -\mu_1 \vec{B} \cdot \vec{L}_1 - \mu_2 \vec{B} \cdot \vec{L}_2.$$

- (a) (5 pts) What values of the total angular momentum might the system attain?
- (b) (15 pts) What are the shifts of the energy due to \mathbf{V}_b ? Assume you needn't worry about mixing with other levels.