PHY 851 – QUANTUM MECHANICS Your Name: ______________________________
MIDTERM I, October 13, 2021

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t,\ \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\ V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\ \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\ |x\rangle e^{ipx/\hbar},\quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2},\quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.$$

$$H = \frac{(\vec{P} - e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x-y):\quad -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A},\quad \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\rm cyclotron} = \frac{eB}{mc},$$

$$e^{A+B} = e^Ae^Be^{-C/2},\quad \text{if } [A,B] = C,\ \text{and } [C,A] = [C,B] = 0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}},\quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),\quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm2i\phi},\quad Y_{\ell-m}(\theta,\phi) = (-1)^mY^*_{\ell m}(\theta,\phi).$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t) = \frac{2\pi}{\hbar}|V_{ni}|^2 \delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r \mathcal{V}(r) e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\rm single}\tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \left|\sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}}\right|^2,$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell+1) i^\ell j_\ell(kr) P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell+1) e^{i\delta_\ell}\sin\delta_\ell \frac{1}{k} P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,\quad \sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell,\quad \delta \approx -ak$$

$$L_\pm|\ell,m\rangle = \sqrt{\ell(\ell+1) - m(m\pm1)}|\ell, m\pm1\rangle,$$

$$C^{\ell,s}_{m_\ell,m_s;JM} = \langle \ell,s,J,M|\ell,s,m_\ell,m_s\rangle,$$

$$\langle\tilde{\beta},J,M|T^k_q|\beta,\ell,m_\ell\rangle = C^{k\ell}_{qm_\ell;JM}\frac{\langle\tilde{\beta},J||T^{(k)}||\beta,\ell,J\rangle}{\sqrt{2J+1}},$$

$$n = \frac{(2s+1)}{(2\pi)^d}\int_{k<k_f} d^dk,\quad d \text{ dimensions},$$

$$\{\Psi_s(\vec{x}), \Psi^\dagger_{s'}(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\delta_{ss'},$$

$$\Psi^\dagger_s(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} a^\dagger_s(\vec{k}),\quad \{\Psi_s(\vec{x}), a^\dagger_\alpha\} = \phi_{\alpha,s}(\vec{x}).$$

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1. (20 pts) At $t = 0$ an electron is in the $|\uparrow\rangle$ (up along the z axis) state, which is represented by

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The evolution is determined by the Hamiltonian,

$$H = A\sigma_z + B\sigma_y.$$

What is the probability the electron will be found in the $|\downarrow\rangle$ state as a function of time?

Solution:

$$\begin{aligned}
e^{-iHt/\hbar} &= e^{-i\sqrt{A^2+B^2}\vec{\sigma}\cdot\hat{n}t/\hbar} \\
\vec{\sigma}\cdot\hat{n} &= \frac{1}{\sqrt{A^2+B^2}}(A\sigma_z + B\sigma_y), \\
e^{-iHt/\hbar} &= \cos(\sqrt{A^2+B^2}t/\hbar) - i\vec{\sigma}\cdot\hat{n}\sin(\sqrt{A^2+B^2}t/\hbar), \\
\langle\downarrow|e^{-iHt/\hbar}|\uparrow\rangle &= \frac{iB}{\sqrt{A^2+B^2}}\sin(\sqrt{A^2+B^2}t/\hbar), \\
\text{Prob} &= \frac{B^2}{A^2+B^2}\sin^2(\sqrt{A^2+B^2}t/\hbar).
\end{aligned}$$

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2. (15 pts) In a one-dimensional world a particle of mass m feels an attractive potential

$$V(x) = \begin{cases} 0, & x < -a \\ -V_0, & -a < x < a \\ 0, & x > a \end{cases}$$

What is the minimum depth of the potential necessary for the number of bound states to be greater or equal to 2.

Solution:
First excited state has one node and is odd, so choose something that goes as $\sin(qx)$ for $x < a$. Next, wave function should barely turn over (slope $\rightarrow 0$ at $x = a$) so choose $qa = \pi/2$, and $E = 0$.

$$\begin{aligned} E &= 0, \\ \frac{\hbar^2 q^2}{2m} &= V_0, \\ q &= \frac{\pi}{2a}, \\ V_0 &= \frac{\hbar^2 \pi^2}{8ma^2}. \end{aligned}$$

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3. A particle of mass m exists in a two-dimensional world and feels a harmonic-oscillator potential,

$$V(x, y) = \frac{1}{2}m\omega^2(x^2 + y^2).$$

(a) (5 pts) What are the eigenenergies?

(b) (10 pts) What are the degeneracies for each level?

Solution:
a) $(N + 1)\hbar\omega, \quad N = 0, 1, 2, 3$
b) $N + 1$

You needn't show your work – credit solely based on writing down correct answer.

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4. (15 pts) A particle of mass $\boldsymbol{m}$ and charge $\boldsymbol{e}$ experiences a magnetic field

$$\vec{B} = B\hat{z},$$

and a weak ($\boldsymbol{E} < \boldsymbol{B}$) electric field

$$\vec{E} = \frac{E}{\sqrt{2}}(\hat{x} + \hat{y}).$$

At $\boldsymbol{t = 0}$ the initial velocity is $v_{0x}\hat{x} + v_{0z}\hat{z}$. Averaging over a long time, what is the velocity (magnitude and direction) of the particle?
You needn't show your work as your grade will be fully determined on your answer alone.

Solution:
In the x-y plane, the magnitude of the drift velocity is E/B and direction is perpendicular to both $\boldsymbol{E}$ and $\boldsymbol{B}$ (i.e. it is parallel to $\vec{E} \times \vec{B}$). Velocity in $\boldsymbol{z}$ direction is constant.

$$\langle \vec{v} \rangle = v_{0z}\hat{z} + \frac{E}{B}\frac{(\hat{x} - \hat{y})}{\sqrt{2}}$$

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5. (10 pts) Evaluate the following matrix element

$$\langle m|(a^\dagger a)^K a^\dagger|n\rangle,$$

where $a^\dagger$ and a are creation and destruction operators respectively.

Solution:

$$\begin{aligned}\langle m|(a^\dagger a)^K a^\dagger|n\rangle &= m^K\langle m|a^\dagger|n\rangle\\ &= m^K\sqrt{n+1}\langle m|n+1\rangle\\ &= m^{K+1/2}\delta_{m,n+1}.\end{aligned}$$

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6. (15pts) Consider a particle of mass m incident on the following one-dimensional potential,

$$V(x) = \begin{cases} \infty, & x < 0 \\ V_0, & 0 < x < a \\ 0, & x > a \end{cases}$$

where $V_0 \to \infty$.

Assume that the incoming wave is e^{-ikx} and the reflected wave is of the form $-e^{2i\delta}e^{ikx}$. Find $\delta(k)$.

Solution:

$$\psi(x > a) = \sin(kr + \delta),$$
$$\text{B.C.} \quad \sin(ka + \delta) = 0,$$
$$\delta = -ka.$$