your name(s)____

QM Subject Exam Review, May 3, 2022

Work in groups of 2 – choose your partner

$$\int_{-\infty}^{\infty} dx\, e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t,\ \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\, V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'), \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\, |x\rangle e^{ipx/\hbar},\ |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2},\ b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.$$

$$H = \frac{(\vec{P} - e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x-y):\quad -\frac{\hbar^2}{2m}\left(\partial_x\psi(x)|_{y+\epsilon} - \partial_x\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A},\ \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\text{cyclotron}} = \frac{eB}{mc},$$

$$e^{A+B} = e^Ae^Be^{-C/2},\ \text{if } [A,B]=C, \text{ and } [C,A]=[C,B]=0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}},\quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),\quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi},\quad Y_{\ell -m}(\theta,\phi) = (-1)^mY^*_{\ell m}(\theta,\phi),$$

$$|N\rangle = |n\rangle - \sum_{m\neq n}|m\rangle\frac{1}{\epsilon_m - \epsilon_n}\langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n}\frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t) = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\mathcal{V}(r)e^{i(\vec{k}_f-\vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k-\epsilon_r)^2+(\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{single}}\tilde{S}(\vec{q}),\ \ \tilde{S}(\vec{q}) = \frac{1}{N}\left|\sum_{\vec{a}}e^{i\vec{q}\cdot\vec{a}}\right|^2 = \sum_{\delta\vec{a}}e^{i\vec{q}\cdot\delta\vec{a}},$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{e^4Z_1^2Z_2^2m^2}{(\hbar k)^4(1-\cos\theta)^2}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}}\left|\frac{1}{e}\int d^3r\ \rho(\vec{r})e^{i\vec{q}\cdot\vec{r}}\right|^2$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell(2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell\frac{1}{k}P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r}f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,\quad \sigma = \frac{4\pi}{k^2}\sum_\ell(2\ell+1)\sin^2\delta_\ell,\ \ \delta \approx -ak$$

$$L_\pm|\ell,m\rangle = \sqrt{\ell(\ell+1)-m(m\pm1)}|\ell,m\pm1\rangle,$$

$$\langle\tilde{\beta},J,M|T_q^k|\beta,\ell,m_\ell\rangle = \langle JM|k,q,\ell,m_\ell\rangle\frac{\langle\tilde{\beta},J||T^{(k)}||\beta,\ell,J\rangle}{\sqrt{2J+1}},$$

$$n = \frac{(2s+1)}{(2\pi)^d}\int_{k<k_f}d^dk,\quad d\text{ dimensions},$$

$$\{\Psi_s(\vec{x}),\Psi_{s'}^\dagger(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\delta_{ss'},$$

$$\Psi_s^\dagger(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}}e^{i\vec{k}\cdot\vec{r}}a_s^\dagger(\vec{k}),\ \ \{\Psi_s(\vec{x}),a_\alpha^\dagger\} = \phi_{\alpha,s}(\vec{x}).$$

1. Consider a $1-$dimensional world. A particle of type A and mass m_A is in the ground state of a harmonic oscillator characterized by frequency ω. It is surrounded by a bath of massless B-particles in a large length L. The massless particles can be absorbed onto the massive A-particles and excite the A particle to an excited state. The probability that a given state for the massless particles is occupied is $f_B(k)$, where l is the wave number of the massless particle. The interaction contribution to the Hamiltonian is

$$V = g\int dx\ [\Phi(x) + \Phi^\dagger(x)]\Psi_A^\dagger(x)\Psi_A(x),$$

where g is a small coupling constant, and the field operators are defined by:

$$\Psi^\dagger(x) = \frac{1}{\sqrt{L}}\sum_q e^{iqx}a^\dagger(\vec{q}),$$
$$\Phi^\dagger(x) = \frac{1}{\sqrt{L}}\sum_k e^{ikx}b^\dagger(\vec{k}).$$

(a) What is the matrix element describing the absorption of a B-particle of wave number k_B that excites the A-particle from the ground state $|A0\rangle$ to the first excited state $A1\rangle$? I.e., find $\mathcal{M}(k_B) = \langle A1|V|A0, k_B\rangle$. Express your answer as an integral – but don't do the integral.

(b) Do the integral and find $\mathcal{M}$. You can use the fact that

$$x = \sqrt{\frac{\hbar}{2m\omega}}(A + A^\dagger),$$

where $A^\dagger$ is the raising operator for the harmonic oscillator, and apply the Baker-Campbell-Hausdorff lemma or use your knowledge of coherent states.

(c) Using Fermi's golden rule, what is the rate at which A particles are excited to the first excited state? Express your answer in terms of $\mathcal{M}$.

2. A fixed charge Ze is spread out uniformly in a spherical ball of radius R, so the density is

$$\rho(r) = \frac{3Ze}{4\pi R^3}\Theta(R - r).$$

A charge e moving with wave number k scatters off the charge density. As a function of the scattering angle θ, what is the differential cross section, $d\sigma/d\Omega$? FYI: Rutherford's cross section is

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{point}} = \frac{4Z^2e^4m^2}{(\hbar q)^4}, \qquad q = |\vec{k}_i - \vec{k}_f|.$$