FINAL (SUBJECT) EXAM,
PHYSICS 852, Spring 2021
April 26, 3:00-6:00 PM

SECRET STUDENT NUMBER:
STUDNUMBER

1. Solutions should be submitted through Gradescope within 20 minutes of the finishing time (6PM), i.e. by 6:20 PM.
2. **DO NOT WRITE YOUR NAME ANYWHERE ON THE EXAM. Your exam has a "secret student number" on each page. Write that number on EACH page of the solutions you submit**.
3. This exam is closed-book and closed-note. You are not permitted to use mathematical software, e.g. Mathematica.
4. This exam has five problems with a summed valued of 100 points.
5. If you have questions during the exam, CALL met at 517-402-3348.

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t,\ \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\ V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\ \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\ |x\rangle e^{ipx/\hbar},\quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2},\quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.$$

$$H = \frac{(\vec{P}-e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x-y):\quad -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A},\quad \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\text{cyclotron}} = \frac{eB}{mc},$$

$$e^{A+B} = e^Ae^Be^{-C/2},\ \text{ if } [A,B] = C,\ \text{and } [C,A] = [C,B] = 0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}},\quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),\quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm2i\phi},\quad Y_{\ell-m}(\theta,\phi) = (-1)^mY^*_{\ell m}(\theta,\phi).$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t) = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\mathcal{V}(r)e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\rm single}\tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \left|\sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}}\right|^2,$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell+1)e^{i\delta_\ell}\sin\delta_\ell\frac{1}{k}P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r}f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,\quad \sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell,\quad \delta \approx -ak$$

$$L_\pm|\ell,m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)}|\ell,m\pm 1\rangle,$$

$$C^{\ell,s}_{m_\ell,m_s;JM} = \langle \ell,s,J,M|\ell,s,m_\ell,m_s\rangle,$$

$$\langle\tilde{\beta},J,M|T^k_q|\beta,\ell,m_\ell\rangle = C^{k\ell}_{qm_\ell;JM}\frac{\langle\tilde{\beta},J||T^{(k)}||\beta,\ell,J\rangle}{\sqrt{2J+1}},$$

$$n = \frac{(2s+1)}{(2\pi)^d}\int_{k<k_f} d^dk,\quad d \text{ dimensions},$$

$$\{\Psi_s(\vec{x}),\Psi^\dagger_{s'}(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\delta_{ss'},$$

$$\Psi^\dagger_s(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}a^\dagger_s(\vec{k}),\quad \{\Psi_s(\vec{x}),a^\dagger_\alpha\} = \phi_{\alpha,s}(\vec{x}).$$

1. A particle of mass m in the first excited state of a one-dimensional harmonic oscillator decays to the ground state. The harmonic oscillator wave functions are:

$$\psi_0(x) = \frac{1}{\pi^{1/4} a^{1/2}} e^{-x^2/2a^2},$$
$$\psi_1(x) = 2^{1/2}(x/a)\psi_0(x),$$

where $a^2 = \hbar/(m\omega)$ and the characteristic energy of the harmonic oscillator is $\hbar\omega$. The decay proceeds through the interaction

$$V = g \int dx \Phi(x) \Psi^\dagger(x) \Psi(x),$$

where g is a coupling constant and $\Psi^\dagger$ and Ψ are field operators for the particle in the harmonic oscillator defined such that

$$\left[\Psi(x), \Psi^\dagger(x')\right] = \delta(x - x').$$

The field operator Φ creates or destroys massless particles and is defined as

$$\Phi = \sum_k \sqrt{\frac{\hbar}{2E_k L}} \left[a_k e^{ikx} + a_k^\dagger e^{-ikx}\right],$$

where L is an arbitrarily large length and $E_k = \hbar c k$.

(a) (10 pts) Write down an integral I such that the matrix element for the transition to the ground state,

$$\mathcal{M} \equiv \langle 0, k | V | 1 \rangle = \frac{1}{\sqrt{L}} I.$$

Here $\langle 0, k|$ refers to the final state, where the massive particle is in the ground state and the massless particle has wavenumber k, while the ket $|1\rangle$ refers to the initial state, where the massive particle is in the first excited state.
<u>DO NOT evaluate the integral!!!</u>

(b) (10 pts) Using Fermi's golden rule, find the decay rate in terms of I, m and ω.

2. A neutron and a proton populate the ground state of a harmonic oscillator. The proton and neutron interact via a spin-spin interaction,

$$\boldsymbol{V_{\text{s.s.}} = -\alpha \vec{S}_p \cdot \vec{S}_n.}$$

Additionally, a magnetic interaction is added,

$$\boldsymbol{V_B = -\beta_n \vec{S}_n \cdot \vec{B} - \beta_p \vec{S}_p \cdot \vec{B}.}$$

Work in the following basis:

$$\boldsymbol{|S=1, M_S=1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |S=1, M_S=-1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},}$$

$$\boldsymbol{|S=1, M_S=0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |S=0, M_S=0\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.}$$

(a) (15 pts) Write the Hamiltonian, $\boldsymbol{V_{\text{s.s.}}} + \boldsymbol{V_B}$, as a $\mathbf{4 \times 4}$ matrix in this basis.

(b) (5 pts) What are the eigen-energies?

3. An external electric field, $\boldsymbol{E_0\hat{x}}$, interacts with a system via the perturbative interaction

$$\boldsymbol{V = -q\vec{E}\cdot\vec{r}.}$$

(a) (10 pts) Circle the matrix elements which might be non-zero.

- $\boldsymbol{\langle\alpha, J=2, M_J=1|V|\beta, J=2, M_J=0\rangle}$
- $\boldsymbol{\langle\alpha, J=2, M_J=1|V|\beta, J=1, M_J=0\rangle}$
- $\boldsymbol{\langle\alpha, J=2, M_J=1|V|\beta, J=0, M_J=0\rangle}$.

(b) (10 pts) Bob and Carol perform a complicated integral to find the matrix element

$$\boldsymbol{I = \langle\alpha, J=1, M=0|z|\beta, J=0, M=0\rangle.}$$

Ted and Alice wish to calculate

$$\boldsymbol{J = \langle\alpha, J=1, M=1|x|\beta, J=0, M=0\rangle.}$$

Express $\boldsymbol{J}$ in terms of $\boldsymbol{I}$ and Clebsch-Gordan coefficients. You need not evaluate the Clebsch-Gordan coefficients.

4. (20 pts) Two identical scattering centers are located at $\boldsymbol{a\hat{x}}$ and $\boldsymbol{-a\hat{x}}$. A beam of wavenumber $\boldsymbol{k}$ is traveling in the $\boldsymbol{\hat{z}}$ direction and interacts weakly with the scatterers. In terms of the polar angle $\boldsymbol{\theta}$ and the azimuthal angle $\boldsymbol{\phi}$, describe the directions for which the scattering amplitude vanishes.

5. A particle of mass m is incident on a spherically symmetric repulsive potential with wave number k

$$V(r) = \begin{cases} V_0, & r < a \\ 0, & r > a \end{cases} .$$

   (a) (15 pts) For incident energies below the barrier, what is the $\ell = 0$ phase shift as a function of k?

   (b) (5 pts) As the incident energy approaches zero, what is the cross section?