FINAL EXAM,

Your Name: ______________________________

PHYSICS 851, FALL 2021

Tuesday, December 14, 12:45 PM

This exam is worth 100 points.

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t,\ \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\ V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\ \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\ |x\rangle e^{ipx/\hbar},\quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2},\quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.$$

$$H = \frac{(\vec{P}-e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x-y):\quad -\frac{\hbar^2}{2m}\left(\partial_x\psi(x)|_{y+\epsilon} - \partial_x\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A},\quad \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\rm cyclotron} = \frac{eB}{mc},$$

$$e^{A+B} = e^Ae^Be^{-C/2},\ \text{ if } [A,B] = C,\ \text{and } [C,A] = [C,B] = 0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}},\quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),\quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm2i\phi},\quad Y_{\ell-m}(\theta,\phi) = (-1)^mY^*_{\ell m}(\theta,\phi).$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt} P_{i\to n}(t) = \frac{2\pi}{\hbar} |V_{ni}|^2 \delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4} \left| \int d^3r \mathcal{V}(r) e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}} \right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)} \frac{4\pi}{k^2} \frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{single}} \tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \frac{1}{N} \left| \sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}} \right|^2 = \sum_{\delta\vec{a}} e^{i\vec{q}\cdot\delta\vec{a}},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} \left| \frac{1}{e} \int d^3r\ \rho(\vec{r}) e^{i\vec{q}\cdot\vec{r}} \right|^2$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell+1) i^\ell j_\ell(kr) P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell+1) e^{i\delta_\ell} \sin\delta_\ell \frac{1}{k} P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,\quad \sigma = \frac{4\pi}{k^2} \sum_\ell (2\ell+1)\sin^2\delta_\ell,\quad \delta \approx -ak$$

$$L_\pm|\ell,m\rangle = \sqrt{\ell(\ell+1) - m(m\pm1)}|\ell, m\pm1\rangle,$$

$$\langle\tilde{\beta}, J, M|T^k_q|\beta,\ell,m_\ell\rangle = \langle JM|k,q,\ell,m_\ell\rangle \frac{\langle\tilde{\beta},J||T^{(k)}||\beta,\ell,J\rangle}{\sqrt{2J+1}},$$

$$n = \frac{(2s+1)}{(2\pi)^d} \int_{k<k_f} d^dk,\quad d \text{ dimensions},$$

$$\{\Psi_s(\vec{x}), \Psi^\dagger_{s'}(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\delta_{ss'},$$

$$\Psi^\dagger_s(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} a^\dagger_s(\vec{k}),\quad \{\Psi_s(\vec{x}), a^\dagger_\alpha\} = \phi_{\alpha,s}(\vec{x}).$$

Your Name: ____________________________

1. (10 pts) Consider spin-vector operators, $\vec{L}$, $\vec{S}$ and $\vec{J} = \vec{L} + \vec{S}$. You can assume $\vec{S}$ operates on intrinsic spin and that $\vec{L}$ describes orbital angular momentum. Circle the operators that commute with J_z?

   - L_x
   - S_z
   - J_x
   - $J_x^2 + J_y^2 + J_z^2$
   - $S_x^2 + S_y^2 + S_z^2$
   - $\vec{L} \cdot \vec{S}$

2. (5 pts) A particle of mass m moving in one-dimension feels the potential

$$V(x) = \begin{cases} \infty, & x < 0 \\ -V_0, & 0 < x < a \\ 0, & x > a \end{cases}$$

   Here V_0 is a positive constant. Assume that m, V_0 and a are initially chosen such that there exists one, but only one, bound state.

   Circle the actions that could increase the number of bound states

   - Increasing the magnitude of V_0
   - Increasing the distance a
   - Increasing the mass m.

3. (5 pts) A particle of mass m moving in one-dimension feels the potential

$$V(x) = \begin{cases} \infty, & x < 0 \\ -V_0\delta(x-a), & x > 0 \end{cases}$$

   Here V_0 is a positive constant. Assume that m, V_0 and a are initially chosen such that there exists one, but only one, bound state.

   Circle the actions that could increase the number of bound states

   - Increasing the magnitude of V_0
   - Increasing the distance a
   - Increasing the mass m.

Your Name: ____________________________

4. A proton and a neutron are in the ground state of a harmonic oscillator. An interaction is added,

$$\boldsymbol{V_{\text{s.s.}} = -\mu_n B S_{nz} - \mu_p B S_{pz},}$$

where $\boldsymbol{S_{nz}}$ and $\boldsymbol{S_{pz}}$ are the spin projection operators for the proton and neutron. We will use $|\boldsymbol{JM}\rangle$ to denote a state of total angular momentum $\boldsymbol{J}$ and projection $\boldsymbol{M}$. At $\boldsymbol{t = 0}$ one is in the state $|\boldsymbol{J = 0, M = 0}\rangle$. Find the probability you are in the following states as a function of time.

(a) (5 pts) $|\boldsymbol{J = 0, M = 0}\rangle$

(b) (5 pts) $|\boldsymbol{J = 1, M = 1}\rangle$

(c) (5 pts) $|\boldsymbol{J = 1, M = 0}\rangle$

(d) (5 pts) $|\boldsymbol{J = 1, M = -1}\rangle$

Advice: To simplify the algebra you might wish to express your work in terms of $\boldsymbol{\beta_n \equiv \mu_n B/2}$ and $\boldsymbol{\beta_p \equiv \mu_p B/2}$, which have units of frequency.

Your Name: ____________________________

(Extra work space for #4)

Your Name: ____________________________

5. A beam of spinless particles of mass m and kinetic energy E is aimed at a spherically symmetric repulsive potential

$$V(r) = \begin{cases} V_0, & r < a \\ 0, & r > a \end{cases}$$

Assume $E < V_0$.

(a) (15 pts) Find the $\ell = 0$ phase shift as a function of the incoming wave number k.

(b) (10 pts) What is the cross section as $k \to 0$?

(Extra work space for #5)

Your Name: ______________________________

6. A particle of mass $\boldsymbol{m}$ moves in a one-dimensional potential

$$\boldsymbol{V(x) = \alpha|x|},$$

where $\boldsymbol{\alpha}$ is a positive constant. **NOTE THE ABSOLUTE VALUE!!!!**
Perform a variational estimate using a Gaussian form for a trial wave function,

$$\boldsymbol{\langle x|b\rangle = \psi_b(x) = \frac{1}{(\pi b^2)^{1/4}} e^{-x^2/2b^2}},$$

where $\boldsymbol{b}$ is the variational parameter. In terms of $\boldsymbol{m, b}$ and $\boldsymbol{\alpha}$,

(a) (10 pts) What is $\boldsymbol{\langle b|KE|b\rangle}$? –the expectation of the kinetic energy.

(b) (10 pts) What is $\boldsymbol{\langle b|V|b\rangle}$? –the expectation of the potential energy.

(c) (15 pts) What is the estimate of the ground state energy?

Advice: The dimensions of $\boldsymbol{\alpha}$ are energy/length if you are checking dimensions.

Your Name: ______________________________

(Extra work space for #6)