FINAL EXAM,
PHYSICS 851, FALL 2019
Noon Friday, December 11, until 5:00 PM Friday, December 18
This exam is worth 100 points.

SECRET STUDENT NUMBER:
STUDNUMBER

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/(2a^2)} = a\sqrt{2\pi},$$

$$H = i\hbar\partial_t,\ \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\ V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\ \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\ |x\rangle e^{ipx/\hbar},\quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2},\quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.$$

$$H = \frac{(\vec{P}-e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x-y):\quad -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A},\quad \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\rm cyclotron} = \frac{eB}{mc},$$

$$e^{A+B} = e^Ae^Be^{-C/2},\ \text{ if } [A,B]=C,\ \text{and } [C,A]=[C,B]=0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}},\quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),\quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi},\quad Y_{\ell -m}(\theta,\phi) = (-1)^mY^*_{\ell m}(\theta,\phi).$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\; n_0(x) = -\frac{\cos x}{x},\; j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\; n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\; n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t) = \frac{2\pi}{\hbar}|V_{ni}|^2 \delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r \mathcal{V}(r) e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{single}} \tilde{S}(\vec{q}), \quad \tilde{S}(\vec{q}) = \left|\sum_{\vec{a}} e^{i\vec{q}\cdot\vec{a}}\right|^2,$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell+1) i^\ell j_\ell(kr) P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\; P_1(x) = x,\; P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell+1) e^{i\delta_\ell}\sin\delta_\ell \frac{1}{k} P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2, \quad \sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell, \quad \delta \approx -ak$$

$$L_\pm|\ell,m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)}|\ell, m\pm 1\rangle,$$

$$C^{\ell,s}_{m_\ell,m_s;JM} = \langle \ell,s,J,M|\ell,s,m_\ell,m_s\rangle,$$

$$\langle\tilde{\beta},J,M|T^k_q|\beta,\ell,m_\ell\rangle = C^{k\ell}_{qm_\ell;JM}\frac{\langle\tilde{\beta},J||T^{(k)}||\beta,\ell,J\rangle}{\sqrt{2J+1}},$$

$$n = \frac{(2s+1)}{(2\pi)^d}\int_{k<k_f} d^dk, \quad d \text{ dimensions},$$

$$\{\Psi_s(\vec{x}),\Psi^\dagger_{s'}(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\delta_{ss'},$$

$$\Psi^\dagger_s(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} a^\dagger_s(\vec{k}), \quad \{\Psi_s(\vec{x}), a^\dagger_\alpha\} = \phi_{\alpha,s}(\vec{x}).$$

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1. (5 pts) Consider three spin operators $\boldsymbol{S_x}, \boldsymbol{S_y}$ and $\boldsymbol{S_z}$. Circle the operators that commute with $\boldsymbol{S_z}$.

   - $\boldsymbol{S_x}$
   - $\boldsymbol{S_z}$
   - $\boldsymbol{S_x^2}$
   - $\boldsymbol{S_z^2}$
   - $\boldsymbol{S_x^2 + S_y^2 + S_z^2}$

2. (5 pts) Consider two sets of spin operators, $\boldsymbol{S_x}, \boldsymbol{S_y}, \boldsymbol{S_z}$ and $\boldsymbol{L_x}, \boldsymbol{L_y}, \boldsymbol{L_z}$. You can assume $\vec{\boldsymbol{S}}$ operates on intrinsic spin and that $\vec{\boldsymbol{L}}$ describes orbital angular momentum. Circle the operators that commute with $\boldsymbol{S_z}$.

   - $\boldsymbol{L_x}$
   - $\boldsymbol{L_z}$
   - $\boldsymbol{L_x^2}$
   - $\boldsymbol{L_z^2}$
   - $\boldsymbol{L_x^2 + L_y^2 + L_z^2}$

3. (5 pts) Now consider the operators $\vec{\boldsymbol{J}} \equiv \vec{\boldsymbol{L}} + \vec{\boldsymbol{S}}$. Circle the operators that commute with $\boldsymbol{S_z}$.

   - $\boldsymbol{J_x}$
   - $\boldsymbol{J_z}$
   - $\boldsymbol{J_x^2}$
   - $\boldsymbol{J_z^2}$
   - $\boldsymbol{J_x^2 + J_y^2 + J_z^2}$

4. (A proton and a neutron are in the ground state of a harmonic oscillator. An interaction is added,

$$V_{\text{s.s.}} = -\alpha \vec{S}_p \cdot \vec{S}_n$$

At $t = 0$ the proton is in a $|\uparrow\rangle$ state and the neutron is in a $|\downarrow\rangle$ state, which we label as $|\uparrow,\downarrow\rangle$. With this labeling the first spin refers to the proton and the second to the neutron.

(a) (15 pts) In the basis above, express $V_{\text{s.s.}}$ as a 4×4 matrix. Use a basis where the states are expressed as

$$|\uparrow,\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\uparrow,\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\downarrow,\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\downarrow,\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(b) (15 pts) Find the probability that the pair is each of the following states as a function of time for $t > 0$.

i. $|\uparrow,\uparrow\rangle$
ii. $|\uparrow,\downarrow\rangle$ (this is the state at $t = 0$)
iii. $|\downarrow,\uparrow\rangle$
iv. $|\downarrow,\downarrow\rangle$

(Extra work space for #4)

5. A beam of spinless particles of mass m and kinetic energy E is aimed at a spherically symmetric repulsive potential

$$V(r) = \begin{cases} V_0, & r < a \\ 0, & r > a \end{cases}$$

Assume $E < V_0$.

(a) (10 pts) Find the $\ell = 0$ phase shift as a function of the incoming wave number k.

(b) (5 pts) What is the cross section as $k \to 0$?

(c) (10 pts) What is the relative probability density for a particle in the wave packet to be at the origin compared to the probability with no potential? I.e. If ρ_0 is the probability density at $r = 0$ in the absence of the potential and ρ is the density with the potential, find ρ/ρ_0.

(Extra work space for #5)

6. A particle of mass m moves in a one-dimensional attractive potential

$$V(x) = -V_0 \exp(-x^2/2a^2).$$

Use a gaussian form for a trial wave function,

$$\langle x|b\rangle = \psi_b(x) = \frac{1}{(\pi b^2)^{1/4}} e^{-x^2/2b^2},$$

where b is the variational parameter.

(a) (10 pts) What is $\langle b|KE|b\rangle$? –the expectation of the kinetic energy.

(b) (10 pts) What is $\langle b|V|b\rangle$? –the expectation of the potential energy.

(c) (10 pts) Find an expression that when solved for b and then plugged into (a) and (b) provides an estimate of the energy. This expression can be a polynomial that needs to be solved for b. (No credit will be given for expressions that are dimensionally inconsistent)

(Extra work space for #6)