FINAL EXAM,
PHYSICS 851, FALL 2019
Thursday, December 12, 7:45-9:45 AM
This exam is worth 150 points

SECRET STUDENT NUMBER:
STUDNUMBER

$$\begin{aligned}
H &= i\hbar\partial_t,\\
\vec{P} &= -i\hbar\nabla,\\
\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\\
U(t,-\infty) &= 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\, V(t')U(t',-\infty),\\
\langle x|x'\rangle = \delta(x-x'), \langle p|p'\rangle &= \frac{1}{2\pi\hbar}\delta(p-p'),\\
|p\rangle = \int dx\, |x\rangle e^{ipx/\hbar}, \quad |x\rangle &= \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},\\
H &= \frac{P^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^\dagger a + 1/2),\\
a^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,\\
\rho(\vec{r},t) &= \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)\\
\vec{j}(\vec{r},t) &= \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t))\\
&\quad - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.\\
H &= \frac{(\vec{P} - e\vec{A}/c)^2}{2m} + e\Phi,\\
\text{For } V &= \beta\delta(x-y),\\
-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{x+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) &= -\beta\psi(y),\\
\vec{E} = -\nabla\Phi - \frac{1}{c}\partial t\vec{A}, \quad \vec{B} &= \nabla\times\vec{A},\\
\omega_{\rm cyclotron} &= \frac{eB}{mc},\\
e^{A+B} &= e^Ae^Be^{-C/2}, \quad \text{if } [A,B]=C, \text{ and } [C,A]=[C,B]=0,\\
Y_{0,0} &= \frac{1}{\sqrt{4\pi}}\\
Y_{1,0} &= \sqrt{\frac{3}{4\pi}}\cos\theta\\
Y_{1,\pm1} &= -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},
\end{aligned}$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t) = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r \mathcal{V}(r) e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma = \frac{(2S_R + 1)}{(2S_1 + 1)(2S_2 + 1)} \frac{4\pi}{k^2} \frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell + 1) i^\ell j_\ell(kr) P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell + 1}} Y_{\ell, m=0}(\theta, \phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2 - 1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell + 1) e^{i\delta_\ell} \sin\delta_\ell \frac{1}{k} P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,$$

$$\sigma = \frac{4\pi}{k^2} \sum_\ell (2\ell + 1)\sin^2\delta_\ell,$$

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/2} = \sqrt{2\pi},$$

$$L_\pm|\ell, m\rangle = \sqrt{\ell(\ell + 1) - m(m \pm 1)}|\ell, m \pm 1\rangle.$$

1. A neutron and proton occupy the ground state of a harmonic oscillator. The particles then feel two additional sources of interaction. First, they have a spin-spin interaction,

$$V_{\text{s.s.}} = \alpha \mathbf{S_n} \cdot \mathbf{S_p},$$

and secondly, they experience an external magnetic field

$$V_b = -\left(\mu_n \mathbf{S_n} + \mu_p \mathbf{S_p}\right) \cdot \vec{B}.$$

Letting J and M reference the total angular momentum and its projection, and letting m_n and m_s reference the projections of the neutron and protons spins,

(a) (10 pts) Circle the operators that commute with the Hamiltonian,

- The magnitude of the total angular momentum, $|\vec{J}|^2 = \hbar^2 J(J+1)$.
- J_z
- $S_z^{(n)}$
- $S_z^{(p)}$

(b) (10 pts) In the J, M basis,

$$|J=1, M=1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ |J=1, M=-1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$|J=1, M=0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |J=0, M=0\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

write the Hamiltonian as a 4×4 matrix.

(c) (5 pts) Find the eigen-energies of the Hamiltonian.

(Extra work space for #1)

(a) only J_z

(b) $$V_{ss} = -\frac{\hbar^2\alpha}{2}\left(J(J+1) - \frac{3}{2}\right)$$

$$= \begin{cases} -\hbar^2\alpha/4, & J = 1 \\ 3\hbar^2\alpha/4, & J = 0 \end{cases}$$

$$= \frac{\hbar^2\alpha}{4}\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 3 \end{pmatrix}$$

$$|J=1, M=1\rangle = |m_n = \tfrac{1}{2}, m_p = \tfrac{1}{2}\rangle$$

$$|J=1, M=-1\rangle = |m_n = -\tfrac{1}{2}, m_p = -\tfrac{1}{2}\rangle$$

$$|J=1, M=0\rangle = \frac{1}{\sqrt{2}}\left(|m_n = \tfrac{1}{2}, m_p = -\tfrac{1}{2}\rangle + |m_n = -\tfrac{1}{2}, m_p = \tfrac{1}{2}\rangle\right)$$

$$|J=0, M=0\rangle = \frac{1}{\sqrt{2}}\left(|m_n = \tfrac{1}{2}, m_p = -\tfrac{1}{2}\rangle - |m_n = -\tfrac{1}{2}, m_p = \tfrac{1}{2}\rangle\right)$$

$$V_b = \frac{\hbar B}{2}\begin{pmatrix} -(\mu_p+\mu_n) & 0 & 0 & 0 \\ 0 & (\mu_p+\mu_n) & 0 & 0 \\ 0 & 0 & 0 & \mu_n-\mu_p \\ 0 & 0 & \mu_n-\mu_p & 0 \end{pmatrix}$$

(c)

$$\mathcal{E}_1 = -\frac{\hbar^2\alpha}{4} - \frac{\mu_p+\mu_n}{2}\hbar B$$

$$\mathcal{E}_2 = -\frac{\hbar^2\alpha}{4} + \frac{\mu_p+\mu_n}{2}\hbar B$$

$$\mathcal{E}_3 = \frac{\hbar^2\alpha}{4} - \sqrt{\left(\frac{\hbar^2\alpha}{2}\right)^2 + \left(\frac{\hbar B}{2}\right)^2(\mu_p-\mu_n)^2}$$

$$\mathcal{E}_4 = \frac{\hbar^2\alpha}{4} + \sqrt{\left(\frac{\hbar^2\alpha}{2}\right)^2 + \left(\frac{\hbar B}{2}\right)^2(\mu_p-\mu_n)^2}$$

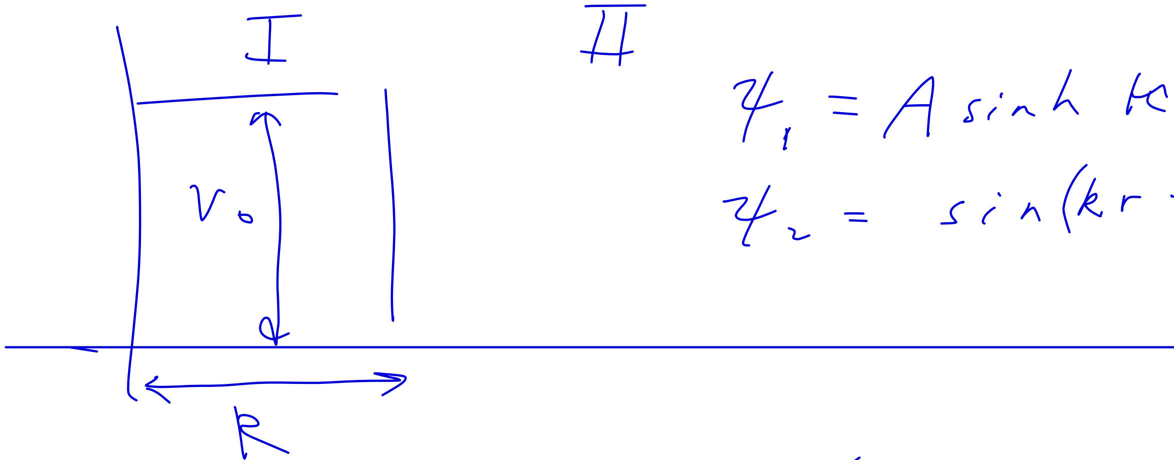
2. A particle of mass m scatters off a target with a spherically symmetric potential,

$$V(r) = V_0 \Theta(R - r).$$

(a) (10 pts) Find the $\ell = 0$ phase shift as a function of the momentum p.

(b) (5 pts) What is the cross-section in the limit that $p \to 0$?

(Extra work space for #2)



$$\psi_1 = A \sinh \kappa r$$
$$\psi_2 = \sin(kr + \delta)$$

(a)

$$A \sinh \kappa R = \sin(kR + \delta)$$
$$\kappa A \cosh \kappa R = k \cos(kR + \delta)$$
$$\frac{1}{\kappa} \tanh \kappa R = \frac{1}{k} \tan(kR + \delta)$$
$$\delta = -kR + \tan^{-1} \frac{k}{\kappa} \tanh(\kappa R)$$
$$-\frac{\hbar^2 \kappa^2}{2m} + V_0 = \frac{\hbar^2 k^2}{2m}$$
$$\kappa = \sqrt{2mV_0/\hbar^2 - k^2}$$

(b)

$$\sigma_{k \to 0} = 4\pi R^2 \left(1 - \frac{\tanh \kappa_0 R}{\kappa_0 R}\right)^2$$
$$\kappa_0 = \sqrt{2mV_0/\hbar^2}$$

3. (15 pts) A particle is initially in the ground state of the two-component system. The initial Hamiltonian is

$$\boldsymbol{H_0 = V_0 \sigma_z.}$$

An interaction is added,

$$\boldsymbol{V(t) = \Theta(t) \beta \sigma_x.}$$

What is the expectation of $\boldsymbol{\sigma_z}$ as a function of time?

(Extra work space for #3)

$$H_0 = V_0 \sigma_z , \qquad V(t) = \beta \sigma_x$$

$$|\psi(t)\rangle = e^{-i V_0 \sigma_z t/\hbar - i\beta \sigma_x t/\hbar} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= e^{-i\sqrt{\frac{V_0^2 + \beta^2}{\hbar^2}}\, \sigma' t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\sigma' = \frac{\beta \sigma_x + V_0 \sigma_z}{\sqrt{\beta^2 + V_0^2}}$$

$$|\psi(t)\rangle = \cos\left(\sqrt{\frac{V_0^2+\beta^2}{\hbar^2}}\, t\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - i \sin\left(\sqrt{\frac{V_0^2+\beta^2}{\hbar^2}}\, t\right) \sigma' \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \left[\cos(\omega t) - i \sin(\omega t) \frac{V_0}{\hbar \omega}\right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} - i \sin \omega t \frac{i\beta}{\hbar \omega} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\omega \equiv \sqrt{\beta^2 + V_0^2}/\hbar$$

$$\langle \sigma_z \rangle = \frac{\beta^2}{\hbar^2 \omega^2} \sin^2 \omega t - \left(\cos^2(\omega t) + \left(\frac{V_0}{\hbar \omega}\right)^2 \sin^2 \omega t \right)$$

$$= -\cos^2 \omega t + \sin^2 \omega t \left(\frac{\beta^2 - V_0^2}{\beta^2 + V_0^2} \right)$$

$\vdots$

4. In one dimension, a particle of type a and mass m is in the ground state of an attractive potential

$$V_0(x) = -\beta\delta(x).$$

A perturbative potential is added,

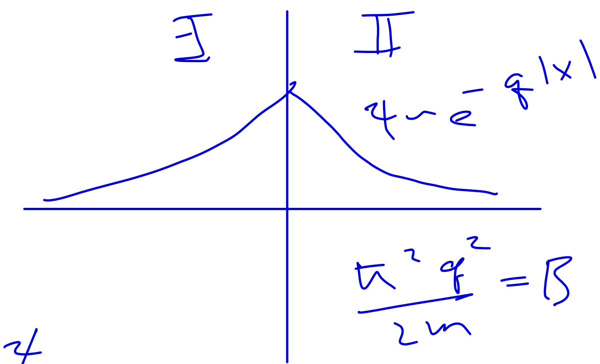
$$V_{ab} = \alpha\cos(\omega t),$$

where α is small and $\hbar\omega$ is larger than the binding energy. This converts the particle to a type b particle, which has the same mass m but does not feel the effects of V_0.

(a) (10 pts) What is the binding energy of the a particle?

(b) (20 pts) What is the decay rate?

(Extra work space for #4)



(a) $$-\frac{\hbar^2}{2m}\left(\psi'_+ - \psi'_-\right) = \beta\,\psi$$

$$\frac{\hbar^2 q}{m} = \beta, \quad B = \frac{\hbar^2}{2m}\left(\frac{m\beta}{\hbar^2}\right)^2$$

$$B = \frac{m\beta^2}{2\hbar^2}$$

(b) $$R = \sum_k \frac{2\pi\alpha^2}{\hbar}\left|\langle 0|k\rangle\right|^2 \delta\left(\varepsilon_k + B - \hbar\omega\right)$$

$$\left|\langle 0|k\rangle\right|^2 = \frac{\alpha^2 q}{L}\left|2\int_0^\infty dx\, e^{-qx}\cos kx\right|$$

$$= \frac{\alpha^2}{L}\left|\int_0^\infty dx\, e^{(ik-q)x} + h.c.\right|^2 q$$

$$= \frac{q}{L}\left|\frac{1}{q-ik} + \frac{1}{q+ik}\right|^2 = \frac{1}{L}\left|\frac{2q}{q^2+k^2}\right|^2$$

$$R = q\frac{\pi\alpha^2}{\hbar}\int\frac{dk}{2\pi}\left|\frac{2q}{q^2+k^2}\right|^2 \delta\left(\frac{\hbar^2k^2}{2m} - (\hbar\omega - B)\right)$$

$$k = \pm\sqrt{\frac{2m}{\hbar^2}\left(\hbar\omega - B\right)}$$

$$= \frac{4mq^3\alpha^2}{k\hbar^3\left(q^2+k^2\right)^2}, \quad q = \frac{m\beta}{\hbar^2}, \quad B = \frac{m\beta^2}{2\hbar^2}$$

5. (20 pts) The cross section for scattering of a particle with momentum $\hbar \boldsymbol{k}$ off a single target is

$$\frac{d\sigma}{d\Omega} = \alpha,$$

which is independent of θ. Now, two targets are placed a distance a apart, separated along the z axis (the same axis as the incident beam moves). At what scattering angles θ does the differential cross section, $d\sigma/d\Omega$, equal zero?

(Extra work space for #5)

$$F(\Omega) = \left| 1 + e^{i q_z a} \right|^2$$

$$q_z = k(1 - \cos\theta)$$

$$q_z a = (2n+1)\pi$$

$$ka(1 - \cos\theta_n) = (2n+1)\pi$$

$$\theta_n = \cos^{-1}\left(ka - (2n+1)\pi\right)$$

$$n = 0, 1, 2, 3 \ldots$$

6. (20 pts) A particle of mass $\boldsymbol{m}$ is in an attractive Coulomb potential, $\boldsymbol{V = -e^2/r}$. Using a Gaussian form,

$$\boldsymbol{\psi = e^{-r^2/2a^2}},$$

as a trial form for the ground state wave function. Provide a variational estimate (upper-bound) for the ground state binding energy.

(Extra work space for #6)

$$E(a) = \left\langle \frac{p^2}{2m} - \frac{e^2}{r} \right\rangle$$

FOR 1-D

$$\langle -\partial_x^2 \rangle = \frac{-1}{a^2\sqrt{\pi}} \int dx \; e^{-x^2/2} \partial_x^2 e^{-x^2/2}$$

$$= \frac{+1}{\sqrt{\pi}\, a^2} \int dx \, e^{-x^2/2} \partial_x \, x \, e^{-x^2/2}$$

$$= \frac{1}{\sqrt{\pi}\, a^2} \int dx \; e^{-x^2/2} (1 - x^2) e^{-x^2/2}$$

3-D

$$= \frac{1}{a^2}\left(1 - \frac{1}{2}\right) = \frac{1}{2a^2}$$

$$E(a) = \frac{3\hbar^2}{4ma^2} - 4\pi \int \frac{r^2 dr}{\pi^{3/2} a^3} e^{-r^2/a^2} \frac{e^2}{r}$$

$$= \frac{3\hbar^2}{4ma^2} - \frac{4e^2}{\pi^{1/2} a} \int_0^\infty x \, dx \, e^{-x^2}$$

$$= \frac{3\hbar^2}{4ma^2} - \frac{2e^2}{\pi^{1/2} a}$$

$$\frac{dE}{da} = 0, \quad a \to \frac{3\hbar^2 \pi^{1/2}}{4me^2}$$

$$E \le \frac{3\hbar^2}{4m} \frac{16 m^2 e^4}{9\hbar^4 \pi} - \frac{2e^2\, 4me^2}{\pi^{1/2} \cdot 3\hbar^2 \pi^{1/2}}$$

$$E \le -\frac{4}{3} \frac{me^4}{\pi \hbar^2}$$

7. A positively charged particle of mass $\boldsymbol{m}$ and charge $\boldsymbol{e}$ is placed in a region with uniform magnetic field $\boldsymbol{B}$ along the $\boldsymbol{z}$ axis.

   (a) (5 pts) Write the vector potential that describes the potential such that $\vec{\boldsymbol{A}}$ is in the $\hat{\boldsymbol{y}}$ direction.

   (b), c) (~~10~~ 5, 5 pts) What ~~are the eigen energies~~ is the g.s. energy, What is general form for all eigenenergies

   (~~c~~ d) (10 pts) An electric field $\boldsymbol{E}$, $|E| < |B|$ is added in the ~~$\boldsymbol{y}$~~ x direction. If the particle is initially at $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{0}$ at time $\boldsymbol{t} = \boldsymbol{0}$ and if the initial velocity is $\vec{\boldsymbol{v}}(\boldsymbol{t} = \boldsymbol{0}) = \boldsymbol{0}$, find its approximate position after a long time $\boldsymbol{t}$. By "approximate", ignore any oscillatory forms to its position vs time.

(Extra work space for #7)

(a) $\vec{A} = -Bx\,\hat{y}$

(b/c) $$H = \frac{p_x^2}{2m} + \frac{\left(p_y - \frac{eB}{c}x\right)^2}{2m} + \frac{p_z^2}{2m}$$

$$= \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2\left(x - \frac{\hbar k_y c}{eB}\right)^2 + \frac{p_z^2}{2m}$$

$$\omega = \frac{eB}{mc}$$

$$E = +\frac{\hbar^2}{2m}k_z^2 + \left(n+\tfrac{1}{2}\right)\hbar\omega, \quad E_0 = \tfrac{1}{2}\hbar\omega$$

(d) $$H = \frac{p_x^2}{2m} + \frac{\left(p_y - \frac{eB}{c}x\right)^2}{2m} - eEx$$

$$= \frac{p_x^2}{2m} + (x - x_0)^2\,\frac{1}{2}m\omega_0^2 + \frac{mc^2}{2}\frac{E^2}{B^2}$$

$$x_0 = \frac{\hbar c k_y}{eB} - \frac{mc^2E}{eB^2}$$

$$\left\langle \frac{\hbar k_y}{m} - \frac{eBx}{c} \right\rangle = v_y = \left\langle \frac{\hbar k_y}{m} - \frac{eBx_0}{c} \right\rangle$$

$$= \frac{eBx_0}{c} - \frac{eBx_0}{c} + \frac{mc^2E}{eB^2}\,\frac{eB}{c}$$

$$= E/B$$