1. Consider the attractive potential,

$$V(x) = \begin{cases} -V_0, & -a < x < a \\ 0, & |x| > a \end{cases}$$

where V_0 is a positive constant.

(a) Solve for the binding energy, B, of the ground state for a particle of mass m. You can leave expression as transcendental equation.

(b) Find expression for B in limit $V_0 >> \hbar^2/ma^2$.

(c) What is the minimum value of V_0 such that there will be multiple bound states?

Solution: a)

$$\begin{aligned} \psi(x) &= \begin{cases} \cos(kx), & |x| < a \\ A\exp(-q|x|), & |x| > a \end{cases} \\ q^2 &= 2mB/\hbar^2, \quad k^2 = 2mV_0/\hbar^2 - q^2, \\ \cos(ka) &= A\exp(-qa), \\ -k\sin(ka) &= -qA\exp(-qa), \\ k\tan(ka) &= q, \\ \sqrt{2m(V_0 - B)}\tan\left[\sqrt{2m(V_0 - B)}a/\hbar^2\right] &= \sqrt{2mB}. \end{aligned}$$

In that limit, it becomes an infinite square well,

$$\begin{aligned} ka &= \pi/2, \\ B &= -V_0 + \frac{\hbar^2 k^2}{2m} = -V_0 + \frac{\hbar^2\pi^2}{8m^2}. \end{aligned}$$

c) solution should be

$$\psi(x) = \begin{cases} \sin(kx), & |x| < a \\ A\exp(-q|x|), & |x| > a \end{cases}$$

BC are

$$\begin{aligned} \sin(ka) &= Ae^{-qa}, \\ k\cos(ka) &= -qAe^{-qa}, \end{aligned}$$

Set $q = 0$ to find limit,

$$\frac{1}{k}\tan(ka) = -\frac{1}{q}.$$

The r.h.s. must be infinite, so $ka = \pi/2$, and

$$B = \frac{\hbar^2\pi^2}{8m^2}.$$

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2. A particle of mass m moves in the positive x direction and is incident on a potential step,

$$V(x) = \begin{cases} 0, & x < 0 \\ -V_0, & x > 0 \end{cases}$$

where V_0 is a positive constant. Find the probability the particle is reflected by the step.

3. A particle of mass m is in the ground state of a harmonic oscillator,

$$V(x, t < 0) = \frac{1}{2} m\omega^2 x^2.$$

At a time $t = 0$, the potential is shifted suddenly to

$$V(x, t > 0) = \frac{1}{2} m\omega^2 (x - a)^2.$$

What is the probability of finding the particle in the ground state of the new potential?

$$\begin{aligned}
\psi_0(x) &= \frac{1}{(\pi b^2)^{1/4}} e^{-x^2/2b^2} \frac{1}{2} m\omega^2 b^2 \qquad\qquad = \hbar\omega/4, \\
b^2 &= \frac{\hbar}{2m\omega}, \\
\psi_1(x) &= \frac{1}{(\pi b^2)^{1/4}} e^{-(x-a)^2/2b^2}, \\
\langle \psi_1 | \psi_0 \rangle &= \frac{1}{\sqrt{(\pi b^2)}} \int dx \; e^{-(x^2+(x-a)^2)/2b^2} \\
&= \frac{1}{\sqrt{(\pi b^2)}} \int dx \; e^{-(x-a/2)^2/b^2} e^{-a^2/4b^2} \\
&= e^{-a^2/4b^2}.
\end{aligned}$$

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4. A particle of mass m and momentum p moves in the positive x direction and is incident on a potential step,

$$V(x) = \begin{cases} 0, & x < 0 \\ V_0, & x > 0 \end{cases}$$

where V_0 is a positive constant. Find the probability the particle is reflected by the step as a function of the momentum p.

5. A particle of mass m is in the ground state of a one-dimensional infinite square well.

$$V(x, t < 0) = \begin{cases} 0, & 0 < x < L \\ \infty, & \text{otherwise} \end{cases}$$

At a time $t = 0$, the potential suddenly disappears. What is the probability density dN/dp that the particle leaves with momentum p? Note that the probability density should be normalize so that $\int dp \; dN/dp = 1$.

6. A massless, $E = cp$, particle moves in an infinite one-dimensional potential well. The energy states as determined by the boundary equations are:

$$E_n = n\hbar\omega_0, \;\; n = 1, 2 \cdots .$$

The system is heated to a temperature T with a single particle in the well. What is the average energy?

Solution:

$$\begin{aligned}
\langle E \rangle &= \frac{\sum_{n=1}^{\infty} n\hbar\omega_0 e^{-n\hbar\omega_0/T}}{\sum_{n=1}^{\infty} e^{-n\hbar_0\omega/T}} \\
&= \frac{-\partial_\beta Z(\beta)}{Z(\beta)}, \quad \beta \equiv 1/T. \\
Z(\beta) &= e^{-\beta\hbar\omega_0} \sum_{n=0}^{\infty} x^n, \quad x \equiv e^{-\beta\hbar\omega_0}, \\
&= e^{-\beta\hbar\omega_0} \frac{1}{1-x} = e^{-\beta\hbar\omega_0} \frac{1}{1 - e^{-\beta\hbar\omega_0}}, \\
\langle E \rangle &= \hbar\omega_0 + \hbar\omega_0 \frac{e^{-\beta\hbar\omega_0}}{1 - e^{-\beta\hbar\omega_0}}.
\end{aligned}$$

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7. The ground state energy of an electron in a Coulomb potential well of a hydrogen atom is -13.6 eV. What is the energy of a μ^- bound to an α particle? Assume the muon is non-relativstic and the that the binding energy is purely due to Coulomb.
The alpha particle is an ionized He-4 nucleus with charge +2. The mass of a muon is $M_\mu = 105$ MeV/c^2, which is 207 times more massive than that of an electron, $M_e = 0.511$. The mass of an α particle is M_α=3727 MeV/c^2 and the mass of a proton or neutron is $M_n = 939$ MeV.

Solution:

$$E = -\frac{Z^2 e^4 M_{\text{red}}}{2\hbar^2},$$

where M_{red} is the reduced mass, $M_{\text{red}} = M_\alpha M_e/(M_\alpha + M_e)$.

$$\begin{aligned}
E &= -\frac{4(M_\alpha M_\mu)(M_p + M_e)}{(M_\alpha + M_\mu) M_p M_e} \cdot 13.6 \text{ eV}, \\
&\approx -800 \cdot 13.6 \text{ eV} \\
&\approx -1.04 \times 10^4 \text{ eV}.
\end{aligned}$$

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8. Consider a two-component system. At $t = 0$ a particle in the state

$$|\psi(t=0)\rangle = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The particle experiences a Hamiltonian

$$H = H_0\sigma_z + V\sigma_x,$$
$$\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_z \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the probability the original state, $|\uparrow\rangle$, is occupied as a function of time.