Practice Problem from § 4 - Electrostatics

CLASSICAL ELECTRODYNAMICS I - PHY841 - Prof. Pratt

Carl E. Fields & Avik Sarkar

SOLUTION

Three charges are located at $-a\hat{\mathbf{y}}$, $+a\hat{\mathbf{y}}$, and $+a\hat{\mathbf{z}}$ with charge $-q$, $-q$, and $+q$, respectively.

`(a)` - Find the electric potential a distance far away from the origin. Consider up to the first two non-zero components of the multipole expansion.

Recall the equation for the monopole term of the electric potential,

$$V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_o}\frac{Q}{r} \;, \tag{1}$$

where Q is the total charge of the configuration. Therefore, we can immediately see that the monopole term should be non-zero and considered in the expansion.
For our system, $Q = (-q) + (-q) + (+q) \equiv -q$, since we are considering discrete points of charge. Therefore the contribution to the electrostatic potential at large r is,

$$V_{\text{mon}}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_o}\frac{q}{r} \;. \tag{2}$$

Next, recall the dipole term for the multipole expansion,

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_o}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \;. \tag{3}$$

First, we must find the dipole moment for our collection of charges, which for discrete point charges, takes the form of,

$$\mathbf{p} = \Sigma_{i=1}^{N} q_i \mathbf{r}'_i \;, \tag{4}$$

where $\mathbf{r}'_i$ is the direction from $\mathcal{O}$ to the $i-$th point charge. For our system, we find

$$\mathbf{p} = \Sigma_{i=1}^{N} q_i \mathbf{r}'_i \equiv (-q)(-a\hat{\mathbf{y}}) + (-q)(+a\hat{\mathbf{y}}) + (q)(+a\hat{\mathbf{z}}) \equiv qa\hat{\mathbf{z}} \;. \tag{5}$$

Therefore, the final dipole term of the electric potential is given as,

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_o}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \equiv \frac{qa}{4\pi\epsilon_o}\frac{\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}}{r^2} \equiv \frac{qa}{4\pi\epsilon_o}\frac{\cos\theta}{r^2} \;, \tag{6}$$

where in the last step, we used the fact that $\hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = \cos\,\theta$.

The total potential is thus written as the sum of the two components,

$$V(r,\theta) \approx V_{\text{mon}}(\mathbf{r}) + V_{\text{dip}}(\mathbf{r}) \approx \frac{q}{4\pi\epsilon_{\text{o}}} \left(-\frac{1}{r} + \frac{a\,\cos\,\theta}{r^2} \right) \; . \tag{7}$$

`(b)` - Using the electric potential from `(a)`, compute the electric field in spherical coordinates. Recall that $\mathbf{E} = -\nabla V$, therefore the components of the electric field are,

$$E_r = \frac{q}{4\pi\epsilon_{\text{o}}} \frac{1}{r^2} - \frac{2a\,\cos\,\theta}{r^3} \; , \tag{8}$$

$$E_\theta = -\frac{q}{4\pi\epsilon_{\text{o}}} \frac{a\,\sin\,\theta}{r^3} \; . \tag{9}$$

Therefore the final electric field for this configuration is found to be,

$$E(r,\theta) = \frac{q}{4\pi\epsilon_{\text{o}}} \left[\left(\frac{1}{r^2} - \frac{2a\,\cos\,\theta}{r^3} \right) \hat{\mathbf{r}} - \frac{a\,\sin\,\theta}{r^3} \hat{\theta} \right] \; . \tag{10}$$