PHY 841 _ HW 4

Solutions

1. Consider two charge densities,

$$\rho_1(\vec{r}) = \frac{3Q_1}{4\pi R_1^3}\Theta(R_1 - |\vec{r}|),$$
$$\rho_2(\vec{r}) = \frac{3Q_2}{4\pi R_2^3}\Theta(R_2 - |\vec{r} - a\hat{x}|).$$

(a) Find the potential energy of the charge distribution when $a \to \infty$.

(b) Find the change in the potential energy for moving from $a = \infty$ to $a > R_1 + R_2$ but finite.

a) You only have PE of each dist. with itself.

$$U_{1\,\text{sphere}} = \frac{1}{2}\int \rho \int d^3r\, V(r) = \frac{3Q_1}{4\pi R_1^3}\cdot 4\pi \int r^2 dr \left[\frac{Q_1}{R_1} + \int_r^{R} \frac{dr'}{r'^2}\frac{r'^3}{R_1^3}Q_1\right]$$

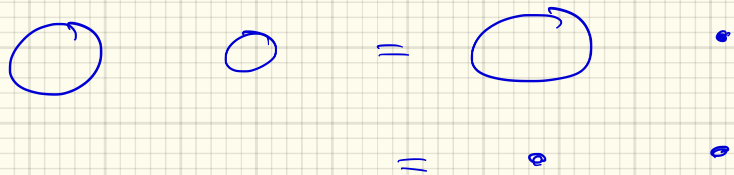
$$= \frac{1}{2}\frac{3Q_1^2}{4\pi R_1^3}\, 4\pi \left\{ \frac{1}{3}R_1^2 + \frac{1}{2R_1^3}\int_0^{R_1} r^2 dr \left[R_1^2 - r^2\right] \right\}$$

$$= \frac{1}{2}\frac{Q_1^2}{R_1} + \frac{3Q_1^2}{4R_1}\left\{\frac{1}{3} - \frac{1}{5}\right\}$$

$$= \frac{Q_1^2}{R_1}\left\{\frac{1}{2} + \frac{3}{30}\right\} = \frac{3}{5}\frac{Q_1^2}{R_1}$$

$$U_{Both} = \frac{3}{5}\left(\frac{Q_1^2}{R_1} + \frac{Q_2^2}{R_2}\right)$$

b)



$$= \frac{Q_1 Q_2}{a}$$

finite.

2. Consider two concentric conducting spherical shells of radius R and $R+a$. The charges on the spheres are Q and $-Q$.

   (a) Calculate the capacitance of the spheres, $C = Q/V$.

   (b) For a potential V, find the electric field as a function of r for all r.

   (c) For a potential V calculate the net energy stored in the electric field.

   (d) Compare this to $CV^2/2$.

a) $C = Q/V$

$$V = Q/R - Q/(R+a)$$

$$C = \frac{1}{\frac{1}{R} - \frac{1}{R+a}} = \frac{R(R+a)}{a}$$

b) For $r < R$, $E = 0$

For $R < r < R+a$,

$E = Q/R^2$ (Gauss's Law)

For $r > R+a$, $E = 0$

c) $$U = \frac{1}{8\pi}\int_R^{R+a} 4\pi r^2 dr \frac{Q^2}{r^4}$$

$$= \frac{Q^2}{2}\left\{\frac{1}{R} - \frac{1}{R+a}\right\}$$

d) By inspection

$U = \frac{Q^2}{2C}$,

$C = Q/V$

$V = Q/C$

$U = \frac{1}{2}CV^2$

3. Show that

$$\boldsymbol{Y_{\ell,\ell}(\theta,\phi) = c_\ell e^{i\ell\phi}\sin^\ell\theta,}$$

$$\boldsymbol{c_\ell = \left[\frac{(-1)^\ell}{2^\ell \ell!}\right]\sqrt{\frac{(2\ell+1)(2\ell)!}{4\pi}}.}$$

is a solution to

$$\boldsymbol{\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta Y(\theta,\phi) + \frac{1}{\sin^2\theta}\partial_\phi^2 Y(\theta,\phi) = -\ell(\ell+1)Y(\theta,\phi),}$$

and that it has unit normalization.

$$Y = C e^{i\ell\phi}\sin^\ell\theta$$

$$\frac{C}{\sin\theta}\partial_\theta \sin\theta\, \partial_\theta \sin^\ell\theta = \frac{\ell C}{\sin\theta}\partial_\theta(\sin^\ell\theta\cos\theta)$$

$$= \ell^2 C \sin^{\ell-2}\theta\cos^2\theta - \ell C\sin^\ell\theta$$

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta Y) = Y\left\{\frac{\ell^2\cos^2\theta}{\sin^2\theta} - \ell\right\}$$

$$\frac{1}{\sin^2\theta}\partial_\phi^2 Y = Y\left(\frac{-\ell^2}{\sin^2\theta}\right)$$

ADD THESE

$$= Y\left\{\frac{-\ell^2+\ell^2\cos^2\theta}{\sin^2\theta} - \ell\right\} = Y(-\ell^2-\ell)$$

$$= -\ell(\ell+1)Y \checkmark$$

4. Show that the expansion in Eq. (4.26) for Legendre polynomials is indeed a solution for

$$\left[(1-x^2)\partial_x^2 - 2x\partial_x + \ell(\ell+1)\right] P_\ell(x) = 0.$$

$$P_\ell(x=\cos\theta) = \frac{1}{2^\ell}\sum_{k=0}^{\ell}\left(\frac{\ell!}{(\ell-k)!k!}\right)^2 (x-1)^{\ell-k}(x+1)^k, \qquad (4.26)$$

$$\partial_x P_\ell = \frac{1}{2^\ell}\sum(\cdots)^2(\ )^{\ell-k}(\ )^k\left[\frac{\ell-k}{x-1}+\frac{k}{x+1}\right]$$

$$-(x+1+x-1)\partial_x P_x = \frac{1}{2^\ell}\sum(\ )^2(\ )^{\ell-k}(\ )^k\left[-\frac{(\ell-k)(x+1)}{(x-1)}-(\ell-k)-k-\frac{k(x-1)}{x+1}\right]$$

$$\partial_x^2 P_\ell = \frac{1}{2^\ell}\sum(\cdots)^2(\ )^{\ell-k}(\ )^k\left[\frac{(\ell-k)(\ell-k-1)}{(x-1)^2}+\frac{k(k-1)}{(x+1)^2}+\frac{2k(\ell-k)}{(x-1)(x+1)}\right]$$

$$(1-x^2)P_\ell = \frac{1}{2^\ell}\sum(\cdots)^2(\ )^{\ell-k}(\ )^k\left[-\frac{(x+1)}{(x-1)}(\ell-k)(\ell-k-1)-\frac{k(k-1)(x-1)}{(x+1)}-2k(\ell-k)\right]$$

$$\left[(1-x^2)\partial_x^2-2x\partial_x+\ell(\ell+1)\right]P_\ell$$

$$=\frac{1}{2^\ell}\sum(\cdots)^2(\ )^{\ell-k}(\ )^k\left\{\frac{(x+1)}{(x-1)}\left[-(\ell-k)(\ell-k-1)-(\ell-k)\right]+\frac{(x-1)}{(x+1)}\left[-k-k(k-1)\right]+\left[-\ell-2k(\ell-k)+\ell(\ell+1)\right]\right.$$

$$=\frac{1}{2^\ell}\sum_k(\cdots)^2(\ )^{\ell-k}(\ )^k\left\{-\frac{(x+1)}{(x-1)}(\ell-k)^2-\frac{(x-1)}{(x+1)}k^2+\ell^2-2k(\ell-k)\right\}$$

$$=\frac{-1}{2^\ell}\sum_k\left(\frac{\ell!}{(\ell-k-1)!\,k!}\right)^2(x-1)^{\ell-k-1}(x+1)^{k+1}$$

$$-\frac{1}{2^\ell}\sum_k\left(\frac{\ell!}{(\ell-k)!\,(k-1)!}\right)^2(x-1)^{\ell-k+1}(x+1)^{k-1}+\frac{1}{2^\ell}\sum_k\left(\frac{\ell!}{(\ell-k)!\,k!}\right)^2(x-1)^{\ell-k}(x+1)^k\cdot(\ell^2-2k(\ell-k))$$

$$=\frac{1}{2^\ell}\sum_k\left(\frac{\ell!}{(\ell-k)!\,k!}\right)^2\left\{-k^2-(\ell-k)^2+\ell^2-2k\ell+2k^2\right\}(x-1)^{\ell-k}(x+1)^k$$

$$=\frac{1}{2^\ell}\sum_k\left(\frac{\ell!}{(\ell-k)!\,k!}\right)^2\left\{-\cancel{2k^2}-\cancel{\ell^2}+\cancel{2k\ell}+\cancel{\ell^2}-\cancel{2k\ell}+\cancel{2k^2}\right\}(x-1)^{\ell-k}(x+1)^k$$

$$=0$$

5. Show that the solution to Laplace's equation in cylindrical coordinates when $k_z = 0$,

$$\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho\right) R(\rho) - \frac{m^2}{\rho^2} R(\rho) = 0,$$

becomes

$$R_m(\rho) = \begin{cases} A\rho^{-m} + B\rho^m, & m \neq 0, \\ C\ln(\rho), & m = 0 \end{cases}.$$

$R = \rho^m$

$\left(\frac{1}{\rho^2} m(m-1) + \frac{m}{\rho^2} - \frac{m^2}{\rho^2}\right) = \emptyset$ ✓

unless $m = \emptyset$

if $m = 0$

$\partial_\rho^2 R = -\frac{1}{\rho} \partial_\rho R$

$R(\rho) =$ constant or $\ln\rho$

$\partial_\rho^2 \ln\rho \stackrel{?}{=} -\frac{1}{\rho} \partial_\rho \ln\rho$

$\partial_\rho \frac{1}{\rho} \stackrel{?}{=} -\frac{1}{\rho^2}$

$-\frac{1}{\rho^2} \stackrel{?}{=} -1/\rho^2$ ✓

6a) $\Phi = x^2 + y^2$

$(\partial_x^2 + \partial_y^2)\,\Phi = 4$

6 b)

$$\frac{1}{r^2}\partial_r\left(r^2\partial_r(r^2\sin^2\theta)\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta(r^2\sin^2\theta)\right)$$

$$= \sin^2\theta\left[\frac{1}{r^2}\partial_r(2r^3)\right] + \frac{1}{\sin\theta}\partial_\theta\left(2\sin^2\theta\cos\theta\right)$$

$$= 6\sin^2\theta + \frac{1}{\sin\theta}\left(4\sin\theta\cos^2\theta - 2\sin^3\theta\right)$$

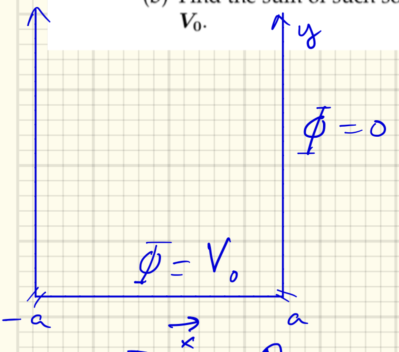
$$= 6\sin^2\theta + \left(4\cos^2\theta - 2\sin^2\theta\right)$$

$$= 4$$

7. Consider ~~the~~ a cavity that extends from $x = -a$ to $x = a$ and from $y = 0$ to $y = \infty$, i.e. it is infinite in the y direction. Assume it is infinite in both directions in the z direction. Along the $y = 0$ boundary, the surface is an insulator kept at a uniform potential V_0, while the boundaries along the $x = \pm a$ surfaces are grounded. Note this means that at the corner of the boundaries the potential is discontinuous, thus one might need to imagine an infinitesimal insulator at the intersection of the boundaries.

(a) Write down general solutions for the system that exponentially die for large y, and that satisfy the B.C. that $\Phi(x = -a, y, z) = \Phi(x = a, y, z) = 0$. For the moment, ignore the B.C. at the $y = 0$ surface.

(b) Find the sum of such solutions from (a) that satisfies the B.C. that $V(x, y = 0, z) = V_0$.



a) $\Phi = A \sin qx \, e^{-qy} + B \cos qx \, e^{-qy}$

b) $V_0 = \sum_n B_n \cos q_n x$, $\Phi = \sum_n B_n \cos q_n x \, e^{-q_n y}$

$q_n a = n\pi/2$

$B_n \neq 0$, only $n = 1, 3, 5, 7 \ldots$

$A_n = 0$ for all n.

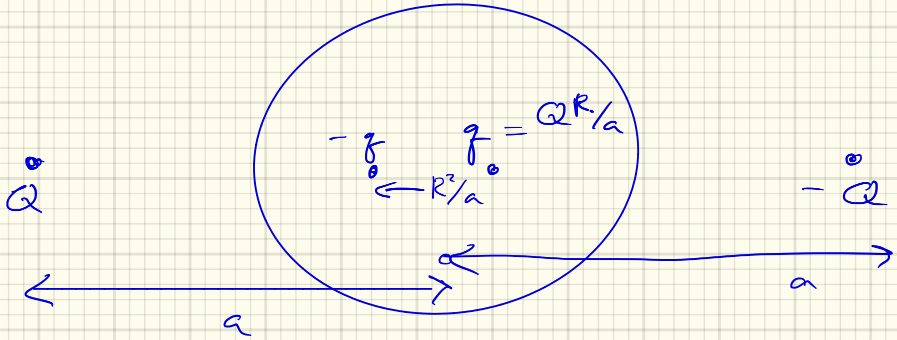
$$B_n = \frac{1}{a}\int_{-a}^{a} V_0 \cos q_n x \, dx$$

$$= \frac{2V_0}{q_n a}(-1)^{(n-1)/2} = \frac{4V_0}{n\pi}(-1)^{(n-1)/2} \quad , n = 1, 3, 5 \ldots$$

$$= 0 \quad , n = 0, 2 \ldots$$

$$\Phi = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots} \frac{(-1)^{(n-1)/2}}{n} \cos\left(\frac{n\pi x}{2}\right) e^{-\frac{n\pi}{2}\frac{y}{a}}$$

8. Consider the solution to a point charge outside a conducting sphere of radius $\boldsymbol{R}$ performed with images from the notes. Consider two point charges, one with charge $\boldsymbol{Q}$ at $-\boldsymbol{a}\hat{\boldsymbol{z}}$, and a second with charge $-\boldsymbol{Q}$ at $\boldsymbol{a}\hat{\boldsymbol{z}}$. As $\boldsymbol{a} \to \infty$, the sphere sees a constant electric field, $\vec{\boldsymbol{E}} = 2\boldsymbol{Q}/\boldsymbol{a}^2\hat{\boldsymbol{z}}$. Find the electric ~~field~~ potential from the two point charges and from the two images in the limit that $\boldsymbol{a}$ and $\boldsymbol{Q}$ both go to infinity in such a way that $2\boldsymbol{Q}/\boldsymbol{a}^2 = \boldsymbol{E}_0$. Compare your solution to that you get from using spherical harmonics.



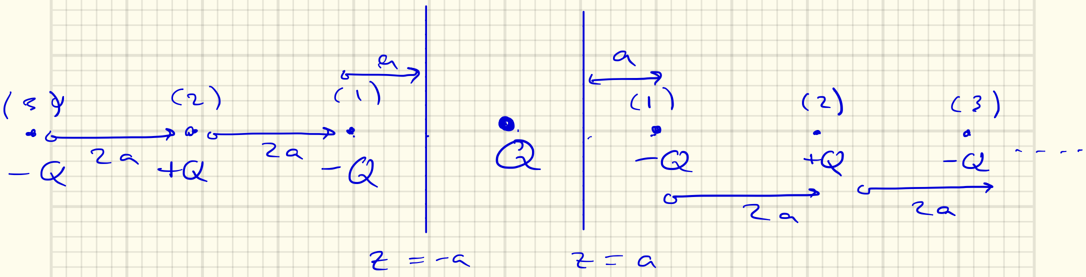
$$\Phi = \frac{Q}{\left[(z+a)^2 + x^2\right]^{1/2}} - \frac{Q}{\left[(z-a)^2 + x^2\right]} - \frac{(QR/a)}{\left[(z+R^2/a)^2 + x^2\right]^{1/2}} + \frac{(QR/a)}{\left[(z-R^2/a)^2 + x^2\right]^{1/2}}$$

$$= \frac{Q}{a} - \frac{Qz}{a^2} - \frac{Q}{a} - \frac{Qz}{a^2} - \frac{(QR/a)}{r} + \frac{(QR/a)}{r}\frac{R^2 z}{r^2 a} + \frac{(QR/a)}{r} + \frac{(QR/a)R^2 z}{r^3 a}$$

$$= -\frac{2Q}{a^2} z + 2\frac{QR^3}{a^2}\frac{z}{r^3}$$

$$= -Ez + \frac{E z R^3}{r^3}$$

9. Consider two infinite conducting planes at $z = a$ and $z = -a$. A point charge Q is replaced at the origin. Find a set of image charges that satisfy the B.C. for $-a < z < a$.



For each plane, there are pairs of equidistant opposite charges that cancel the potential.

10. Using the solution for a conducting cylinder in a constant field from the notes, show that the electric field is perpendicular to the surface at $\rho = R$.

From the notes the E-field is

$$E_x = E_0 - \frac{E_0 R^2}{\rho^2} + \frac{E_0 x^2 R^2}{\rho^4},$$
$$E_y = \frac{E_0 xy R^2}{\rho^4}.$$

At $\rho = R$,

$$E_x = E_0 - E_0 + E_0 \frac{x^2}{R^2}$$

$$E_y = E_0 \frac{xy}{R^2}$$

$$\vec{E}\Big|_{\rho=R} = \frac{E_0 x}{R^2}(x\hat{x} + y\hat{y})$$

which is perpendicular to the surface, i.e. $\parallel$ to $\vec{r}$

11. Consider a sphere of radius R centered at the origin, The surface of the potential is $V(\cos\theta)$.

(a) In spherical coordinates, using the azimuthal symmetry, the potential at $r = R$ can be written as

$$\Phi(r = R, \cos\theta) = \sum_{\ell} a_\ell P_\ell(\cos\theta).$$

Find C_ℓ in the expression for a_ℓ of the form,

$$a_\ell = C_\ell \int_{-1}^{1} dx\, \Phi(r = R, x) P_\ell(x).$$

Here are some identities you might find useful:

$$\begin{aligned}
P_\ell(x=1) &= 1, \\
\sum_\ell (2\ell+1) P_\ell(x) P_\ell(x') &= 2\delta(x-x'), \\
\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) &= \frac{2}{2\ell+1}\delta_{\ell\ell'}, \\
\sum_\ell (2\ell+1) P_\ell(x) P_\ell(x') &= 2\delta(x-x'), \\
(2\ell+1) P_\ell(x) &= \frac{d}{dx}\left[P_{\ell+1}(x) - P_{\ell-1}(x)\right], \\
(\ell+1) P_{\ell+1}(x) &= (2\ell+1) x P_\ell(x) - \ell P_{\ell-1}(x), \\
P_\ell(x) &= \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell \ \text{(Rodriguez formula)}.
\end{aligned}$$

(b) Find a_ℓ for all ℓ for the potential

$$\Phi(r = R, \cos\theta) = V_0 \cos(2\theta).$$

Assuming the inside of the sphere is empty, write the potential $\Phi(\vec{r})$ for all $\vec{r}$.

a) $V(x) = \sum_\ell a_\ell P_\ell(x)$

$$\int V(x) P_{\ell'}(x)\,dx = \sum_\ell a_\ell \int P_\ell(x) P_{\ell'}(x)\,dx$$

$$= \frac{2}{(2\ell+1)} \sum_\ell a_\ell \delta_{\ell\ell'} = \frac{2}{(2\ell+1)} a_\ell$$

$$a_\ell = \frac{(2\ell+1)}{2} \int_{-1}^{1} dx\, P_\ell(x) V(x) \quad , \quad \boxed{C_\ell = \frac{2\ell+1}{2}}$$

b) $V = V_0 \cos 2\theta = V_0 (2x^2 - 1)$

$P_0 = 1$, $P_2 = \frac{3}{2}x^2 - 1/2$

$$V = V_0 \cdot \left\{ \frac{4}{3} P_2(x) - \frac{1}{3} P_0(x) \right\}.$$

$$a_0 = \frac{V_0}{2} \int_{-1}^{1} dx \left(-\frac{1}{3} P_0(x) \right) P_0(x) = -\frac{V_0}{3}$$

$$a_2 = V_0 \frac{3}{2}\frac{4}{3} \int_{-1}^{1} dx\, P_2(x) \cdot P_2(x)$$

$$= 4V_0 \int_0^1 dx \left(\frac{3}{2}x^2 - \frac{1}{2} \right)^2 = V_0 \int_0^1 dx \left[9x^4 - 6x^2 + 1 \right]$$

$$= V_0 \left\{ \frac{9}{5} - 2 + 1 \right\} = \frac{4}{5} V_0$$

$$V(\vec{r}) = \sum_\ell \frac{1}{r^{\ell+1}} b_\ell P_\ell(\cos\theta)$$

$$\frac{b_\ell}{R^{\ell+1}} = a_\ell$$

$$V(\vec{r}) = -\frac{R}{3r} V_0 + \frac{4}{5} \frac{R^3}{r^3} V_0 \left(\frac{3}{2}\cos^2\theta - \frac{1}{2} \right)$$

12. Like the previous problem, but with the potential

$$\Phi(r = R, \cos\theta) = \begin{cases} V_0, & \cos\theta > 0 \\ -V_0 & \cos\theta < 0 \end{cases}$$

(a) Using the identities from the previous problem, show that for this potential

$$a_\ell = V_0 P_{\ell-1}(x = 0)\frac{(2\ell+1)}{(\ell+1)}.$$

(b) Again, using the identities above, show that

$$\begin{aligned} P_{\ell+1}(x=0) &= -\frac{\ell}{(\ell+1)} P_{\ell-1}(x=0), \\ P_{\ell-1}(x=0) &= -\frac{(\ell-2)}{(\ell-1)} P_{\ell-3}(x=0). \end{aligned}$$

(c) Putting these together, show that

$$\begin{aligned} a_\ell &= -a_{\ell-2}\frac{(2\ell+1)(\ell-2)}{(\ell+1)(2\ell-3)}, \\ a_1 &= 3V_0/2, \quad a_{(\text{even})} = 0. \end{aligned}$$

(d) To test your answer, write a short program to calculate $\Phi(r = R)$ and see whether it matches the expectation.

a)

$$a_\ell = \phi \quad \ell = \text{even}$$

$$\begin{aligned} a_{\ell=\text{odd}} &= (2\ell+1)V_0 \int_0^1 dx \; P_\ell(x) \\ &= V_0 \int_0^1 dx \frac{d}{dx}\left[P_{\ell+1} - P_{\ell-1}\right] \\ &= -V_0 \left\{ P_{\ell+1}(0) - P_{\ell-1}(0) \right\} \\ &= -V_0 \left\{ \frac{1}{(\ell+1)}\left[-\ell P_{\ell-1}(0)\right] - P_{\ell-1}(0) \right\} \\ &= +V_0 P_{\ell-1}(0) \cdot \frac{(2\ell+1)}{\ell+1} \end{aligned}$$

b)

$$P_{\ell+1}(x=0) = \frac{-\ell}{\ell+1} P_{\ell-1}(x=0)$$

$$P_{\ell-1}(x=0) = \frac{-(\ell-2)}{(\ell-1)} P_{\ell-3}(x=0)$$

c)

$$a_\ell = V_0 \frac{(2\ell+1)}{(\ell+1)} \cdot (-1) \frac{(\ell-2)}{(\ell-1)} P_{\ell-3}(0)$$

$$a_{\ell-2} = V_0 P_{\ell-3}(0) \frac{(2\ell-3)}{(\ell-1)} \qquad P_{\ell-3}(0) = \frac{a_{\ell-2}}{V_0} \cdot \frac{(\ell-1)}{(2\ell-3)}$$

$$a_\ell = -a_{\ell-2} \cdot \frac{(2\ell+1)}{\ell+1} \frac{(\ell-2)}{\cancel{(\ell-1)}} \frac{\cancel{(\ell-1)}}{(2\ell-3)} = -a_{\ell-2} \frac{(2\ell+1)}{(\ell+1)} \frac{(\ell-2)}{(2\ell-3)}$$

$$a_1 = 1.5\, V_0$$