Statistical Mechanics Subject Exam, Dec. 9, 2008

SECRET STUDENT NUMBER: STUDNUMBER

FUN FACTS TO KNOW AND TELL

$$\begin{aligned}
\int_0^\infty dx\ \frac{x^{n-1}}{e^x - 1} &= \Gamma(n)\zeta(n), \qquad \int_0^\infty dx\ \frac{x^{n-1}}{e^x + 1} = \Gamma(n)\zeta(n)\left[1 - (1/2)^{n-1}\right], \\
\zeta(n) &\equiv \sum_{m=1}^{\infty} m^{-n}, \quad \Gamma(n) \equiv (n-1)!, \\
\zeta(3/2) &= 2.612375..., \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(3) = 1.20205..., \quad \zeta(4) = \frac{\pi^4}{90}, \\
& \int_{-\infty}^{\infty} dx\ e^{-x^2/2} = \sqrt{2\pi}, \qquad \int_0^\infty dx\ x^n e^{-x} = n!
\end{aligned}$$

LONG ANSWER SECTION

1. (10 pts) Consider a finite system where the thermodynamics are determined by the temperature and some other variable x, which can be determined by dialing a knob. Beginning with the relation,

$$TdS|_x = dE|_x, \quad \text{i.e.} \quad \left.\frac{\partial E}{\partial S}\right|_x = T,$$

and the definition of the Helmholtz free energy, $F \equiv E - TS$, show that

$$\left.\frac{\partial E}{\partial x}\right|_S = \left.\frac{\partial F}{\partial x}\right|_T$$

Extra work space for #1

2. A two-dimensional gas of non-relativisitic spin-1/2 fermions of mass m is confined within an area A is thermalized according to a chemical potential μ and temperature T, i.e., the phase space occupancy is

$$f(p) = \frac{e^{-(E_p-\mu)/T}}{1+e^{-(E_p-\mu)/T}}.$$

Originally, the temperature is $T = 0$ and the chemical potential is μ_0.

(a) (5 pts) In terms of the m, A and μ_0 find the average number of particles N when $T = 0$.

(b) (5 pts) Find the single-particle density of states D (number of states per energy) in terms of m, A and the single-particle energy ϵ.

(c) (10 pts) Assuming the density is held constant while the temperature is slighty raised, find the change in the chemical potential to second order in the temperature. Express your answer in the form,

$$\mu = \mu_0 + \alpha_1 T + \alpha_2 T^2,$$

expressing α_1 and α_2 in terms of μ_0 and m evaluated at the Fermi surface.

Extra work space for #2

3. (15 pts) The partition function for a two-dimensional spin system in a magnetic field B is

$$Z = \mathrm{Tr}\exp\left(-\beta H_0 + \beta\mu_b B\int d^2r\ m(r)\right).$$

After some calculations, one finds the spin-spin correlation function for zero field to be,

$$\langle (m(r=0) - \langle m\rangle)(m(r) - \langle m\rangle)\rangle = \alpha(T)e^{-r/\ell}.$$

Calculate the susceptibility,

$$\chi \equiv \frac{d\langle m\rangle}{dB},$$

and express your answer in terms of T, α, ℓ and μ_b.

Extra work space for #3

SHORT ANSWER SECTION

4. (2 pts each) A system has five energy states. The ground state is doubly degenerate, with energy $\epsilon_0 = 0$. The remaining three states are triply degenerate with energy E.

   (a) What is the average energy when $T = 0$? ____________
   (b) What is the entropy when $T = 0$? ____________
   (c) What is the average energy when $T >> \epsilon$? ____________
   (d) What is the entropy when $T >> \epsilon$? ____________

5. (2 pts each) Pick the appropriate number of dimensions d for each of the following:

   (a) For Bose condensation of a gas of MASSLESS non-interacting particles to occur, d must be greater than __________.
   (b) For Bose condensation of a gas of a MASSIVE (non-relativistic) non-interacting particles to occur, d must be greater than __________.

6. Imagine a box with volume V in which there is a dilute gas of N ^{4}He atoms. The volume in the box is doubled while keeping N fixed. The box is well insulated so that no heat enters the box. (2 pts each)

   (a) The volume is doubled by suddenly removing a partition that initially confined the particles to one side, then one waits until the particles re-thermalize within the volume. The temperature will: (Circle one)
      - decrease
      - stay the same
      - increase

   (b) The volume is doubled by suddenly removing a partition that initially confined the particles to one side, then one waits until the particles re-thermalize within the volume. The net entropy will: (Circle one)
      - decrease
      - stay the same
      - increase

   (c) The volume is doubled by slowly sliding the partition that initially confined the particles to one side. The temperature will: (Circle one)
      - decrease
      - stay the same
      - increase

   (d) The volume is doubled by slowly sliding the partition that initially confined the particles to one side. The net entropy will: (Circle one)
      - decrease
      - stay the same
      - increase

7. (3 pts) Two phase transitions will be of the same universality class if they have: (circle one)

   - The same order parameters
   - The same critical exponents
   - The same Goldstone bosons
   - The same dimensionality

8. (2 pts each) Consider a one-dimensional Ising model at temperature $T > 0$. Label each of the following as true or false.

   (a) In the exact solution there is no phase transition. ____________

   (b) In the mean-field solution there is no phase transition. ____________

   (c) In the mean-field solution, the critical exponents are the same as they would be for a two-dimensional model. ____________

9. (3 pts each) Consider a two-dimensional square lattice of N coupled oscillators. While the plane is in the $x - y$ plane, the oscillators move only in the $\pm z$ direction. Sound waves move with velocity c_s. Assume the lattice is one atom thick in the z direction and infinitely long in the x and y direction. Let C/N refer to the specific heat per oscillator.

   (a) As $T \to 0$, the specific heat from phonons behaves as $C \sim T^n$. What is n? ____________

   (b) What is C/N as $T \to \infty$? ____________