

DO NOT WRITE YOUR NAME OR STUDENT NUMBER ON ANY SHEET!

FUN FACTS TO KNOW AND TELL

$$\int_0^\infty dx \frac{x^{n-1}}{e^x - 1} = \Gamma(n)\zeta(n), \quad \int_0^\infty dx \frac{x^{n-1}}{e^x + 1} = \Gamma(n)\zeta(n) \left[1 - (1/2)^{n-1}\right],$$
$$\zeta(n) \equiv \sum_{m=1}^{\infty} m^{-n}, \quad \Gamma(n) \equiv (n-1)!,$$
$$\zeta(3/2) = 2.612375\dots, \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(3) = 1.20205\dots, \quad \zeta(4) = \frac{\pi^4}{90},$$
$$\int_{-\infty}^{\infty} dx e^{-x^2/2} = \sqrt{2\pi}, \quad \int_0^\infty dx x^n e^{-x} = n!$$

LONG ANSWER SECTION

1. Consider a large two-dimensional array of N coupled harmonic oscillators in area A that lie in the $x - y$ plane when at rest. The oscillator's movement is also confined to the $x - y$ plane. In the absence of the coupling, each oscillator has a fundamental frequency ω_0 . After coupling both the longitudinal and transverse sound modes have a speed c_s .

(a) (10 pts) Solve for the Debye frequency, ω_D , in terms of ω_0 , N , A and c_s .

(b) (10 pts) For $T \ll \hbar\omega_D$, find the specific heat per unit area,

$$C \equiv (1/A)d\langle E \rangle/dT.$$

(c) (5 pts) What is $C(T \rightarrow \infty)$?

SECRET STUDENT NUMBER: STUDNUMBER

Extra workspace for #1

SECRET STUDENT NUMBER: STUDNUMBER

2. (10 pts) N ink molecules are placed in a vessel of length L , $0 < x < L$. The molecules diffuse according to a diffusion constant D , i.e., the density satisfies the diffusion equation,

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2}.$$

At $t = 0$, the density has the form

$$\rho(x, t = 0) = A \sin(\pi x / L),$$

Assuming the molecules **adsorb** at the boundaries (stick), find $\rho(x, t > 0)$.

3. A two-dimensional gas of non-relativistic spin-1/2 fermions of mass m are confined within an area A and are thermalized according to a chemical potential μ and a temperature T . Originally, the temperature is $T = 0$ and the chemical potential is $\mu > 0$.
- (a) (5 pts) In terms of m and A , find the density of single-particle states, $D(\epsilon)$.
 - (b) (5 pts) In terms of A , m and μ , find the average number of particles when $T = 0$.
 - (c) (10 pts) Assuming μ is held constant while the temperature is raised slightly, find the change in the average number of particles to second order in the temperature. Express your answer in the form: (find b_1 and b_2)

$$N = N_0 + b_1 T + b_2 T^2.$$

SECRET STUDENT NUMBER: STUDNUMBER

Extra work space for #3

4. Suppose the average energy \bar{E} and the average number of particles \bar{N} in a one-dimensional system of extent L are given as a function of T , L and $\alpha \equiv -\mu/T$. Further assume that L is much larger than any microscopic scale or correlation length of the system.

(a) (10 pts) Derive an expression for the specific heat per unit length,

$$C \equiv \left. \frac{1}{L} \frac{\partial \bar{E}}{\partial T} \right|_N,$$

in terms of $T, L, \bar{E}, \bar{N}, \partial_T \bar{E}|_\alpha, \partial_\alpha \bar{E}|_T, \partial_T \bar{N}|_\alpha$ and $\partial_\alpha \bar{N}|_T$.

(b) (10 pts) Assume the correlations in the system are sufficiently local they can be expressed in terms of delta functions,

$$\begin{aligned} \langle \Delta \rho(0) \Delta \rho(x) \rangle|_{\alpha, T} &= A_{\rho\rho} \delta(x), \\ \langle \Delta \epsilon(0) \Delta \epsilon(x) \rangle|_{\alpha, T} &= A_{\epsilon\epsilon} \delta(x), \\ \langle \Delta \epsilon(0) \Delta \rho(x) \rangle|_{\alpha, T} &= A_{\epsilon\rho} \delta(x), \end{aligned}$$

where ϵ and ρ are the energy density and number density respectively. Express C in terms of $T, \alpha, \bar{N}, \bar{E}, A_{\rho\rho}, A_{\epsilon\epsilon}$ and $A_{\rho\epsilon}$.

SECRET STUDENT NUMBER: STUDNUMBER

Extra work space for #4

SHORT ANSWER SECTION

5. (2 pts each) Three identical spin-zero particles can each occupy one of two single-particle energy levels, 0 and ϵ .
- (a) What is the average energy when $T = 0$? _____
- (b) What is the entropy when $T = 0$? _____
- (c) What is the average energy when $T \gg \epsilon$? _____
- (d) What is the entropy when $T \gg \epsilon$? _____
6. (3 pts) If you read an article where the authors maximize the pressure to solve for an order parameter ϕ , which quantities can you assume were fixed as ϕ was varied? Circle all that are true.
- (a) entropy
- (b) temperature
- (c) particle number
- (d) density
- (e) chemical potential
- (f) energy density
7. (4 pts) A box of volume $2V$ is initially set up with N indistinguishable particles partitioned on one side and N of the same species of particles on the right side. Both sides initially have volume V , are at the same temperature T , and behave like ideal gases. The partition is an excellent conductor of heat so heat can readily flow from one partition to the other, but the overall system is extremely well insulated from the outside, and no heat leaves the system. Beginning at $t = 0$, the partition is slowly (compared to rate at which heat moves from one side to the other) moved until the left side has volume $3V/2$ and the right side has volume $V/2$. Circle all that are true:
- The pressures remain equal to one another, $P_L = P_R$
 - The temperatures remain equal to one another, $T_L = T_R$
 - The entropy densities remain equal to one another, $s_L = s_R$
 - The overall entropy remains constant, $S_L + S_R = \text{constant}$
 - The overall energy remains constant, $E_L + E_R = \text{constant}$

