SECRET STUDENT NUMBER: STUDNUMBER

FUN FACTS TO KNOW AND TELL

$$\int_0^\infty dx\ \frac{x^{n-1}}{e^x-1} = \Gamma(n)\zeta(n), \qquad \int_0^\infty dx\ \frac{x^{n-1}}{e^x+1} = \Gamma(n)\zeta(n)\left[1-(1/2)^{n-1}\right],$$

$$\zeta(n) \equiv \sum_{m=1}^{\infty} m^{-n}, \quad \Gamma(n) \equiv (n-1)!,$$

$$\zeta(3/2) = 2.612375..., \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(3) = 1.20205..., \quad \zeta(4) = \frac{\pi^4}{90},$$

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/2} = \sqrt{2\pi}, \qquad \int_0^\infty dx\ x^n e^{-x} = n!$$

LONG ANSWER SECTION

1. (10 pts) Given

$$TdS = dE + PdV - \mu dN,$$

show that

$$T\left.\frac{\partial N}{\partial T}\right|_{V,\mu/T} = \left.\frac{\partial E}{\partial \mu}\right|_{T,V}.$$

Extra work space for #1

2. Consider a one-dimensional system of fixed number N in a fixed length $L \to \infty$, with the partition function given by:

$$Z = \text{Tr } \exp\{-\beta H\}, \quad H = \int h(x)dx,$$

where $h(x)$ is the Hamiltonian density operator. After great effort, the correlation function is calculated,

$$\Gamma(x - x') = \langle (h(x) - \epsilon)(h(x') - \epsilon) \rangle = \Gamma_0 e^{-|x-x'|/\lambda},$$

where $\langle A \rangle$ denotes a thermal average, $\langle A \rangle = \text{Tr } Ae^{-\beta H}/\text{Tr } e^{-\beta H}$, and $\epsilon = \langle h(x) \rangle$ is the average energy density.

(a) (5 pts) Calculate the fluctuation of the total energy,

$$\sigma_E^2 = \left\langle (H - E)^2 \right\rangle, \quad E = \epsilon L,$$

in terms of Γ_0, T, N, λ and L.

(b) (10 pts) In terms of the same variables, what is the specific heat,

$$C = \frac{dE}{dT}.$$

Extra work space for #2

3. (15 pts) You have discovered a new species of spin-zero bosons called weirdons which you can confine to a one-dimensional motion. Weirdons interact very weakly with one another and are unusual because of their dispersion relation,

$$\epsilon_p = A\sqrt{p}.$$

(A normal dispersion relation would be $\epsilon_p = p^2/2m$).

You have a gas of weirdons with density (number per unit length) ρ. To what temperature must you cool the weirdons to get Bose condensation?

Extra work space for #3

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SHORT ANSWER SECTION

4. (2 pts each) Two non-interacting electrons occupy two single-particle energy levels ϵ and $-\epsilon$.
   (a) What is the average energy when $T = 0$? ____________
   (b) What is the entropy when $T = 0$? ____________
   (c) What is the average energy when $T >> \epsilon$? ____________
   (d) What is the entropy when $T >> \epsilon$? ____________

5. (2 pts each, 8 pts total) You are performing a calculation to determine the thermodynamically optimum concentration x of some chemical species. For each question circle either **maximize** or **minimize**, and one of the quantities S, F, P, G, where S is the entropy, $F = E - TS$ is the Helmholtz free energy, P is the pressure and $G = \mu N$ is the Gibb's free energy.
   (a) If the system is kept at fixed temperature, pressure and particle number (the volume can adjust to match the desired pressure), you should **(maximize/minimize)** the quantity (S, F, P, G).
   (b) If the system is kept at fixed volume, temperature and chemical potential (there is a particle bath and a heat bath), you should **(maximize/minimize)** the quantity (S, F, P, G).
   (c) If the system has fixed particle number, volume and temperature (heat bath only), you should **(maximize/minimize)** the quantity (S, F, P, G).
   (d) If the system is isolated at fixed energy, particle number and volume, you should **(maximize/minimize)** the quantity (S, F, P, G).

6. (3 pts each) Consider a one-dimensional array of N coupled oscillators lined up along the z direction. The oscillators are allowed to move only in the x and y directions. Transverse waves move with velocity c_s. Let C/N refer to the specific heat per oscillator.
   (a) As $T \to 0$, the specific heat from phonons behaves as $C \sim T^n$. What is n? ____________
   (b) What is C/N as $T \to \infty$? ____________

7. (2 pts each) Consider a one-dimensional Ising model at temperature $T > 0$. Label each of the following as true or false.

   (a) In the exact solution there is no phase transition. ____________

   (b) In the mean-field solution there is no phase transition. ____________

   (c) In the mean-field solution, the critical exponents are the same as they would be for a two-dimensional model. ____________

8. (2 pts each) Graph several isotherms on a P vs. V graph illustrating the characteristics of a liquid gas phase transition. The graph should include:

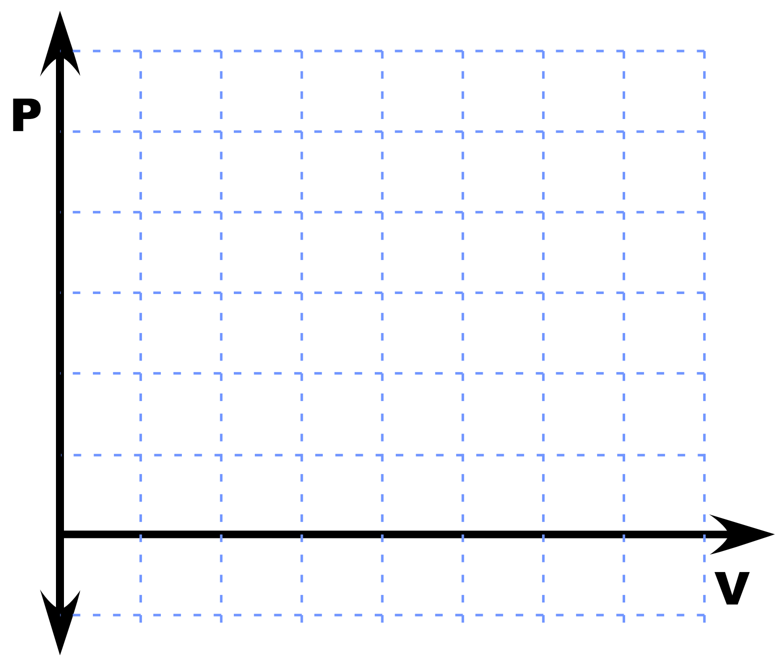
   (a) An isotherm with $T > T_c$.

   (b) An isotherm with $T = T_c$.

   (c) An isotherm with $T < T_c$.

   (d) Label the critical point.

   (e) For the isotherm with $T < T_c$, label the coexistence points.



9. (3 pts) Two phase transitions will be of the same universality class if they have: (circle one)

   - The same order parameters
   - The same critical exponents
   - The same Goldstone bosons
   - The same dimensionality

10. (6pts) Consider a **three-dimensional** Ising model, where each spin can have $\sigma = \pm 1$, and nearest neighbor spins experience an attractive interaction, $H_{nn} = -J\sum_{ij}\sigma_i\sigma_j$. Each spin also experiences an interaction with an external field, $H_B = -\mu B\sum_i \sigma_i$. Plot $\langle\sigma\rangle$ as a function of the magnetic field B (qualitatively) for isotherms with:

    (a) $T = T_c/2$

    (b) $T = 2T_c$