Section 4.11-4.12

by Reyes Rivera and Fan

Molecules are placed in a liquid at a time $t = 0$ and diffuse according to a diffusion constant D, i.e., the density of molecules satisfy the diffusion equation,

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2}$$

(a) Suppose at $t = 0$ we have:

$$\rho(x, 0) = \delta(x)$$

Find the value of A(t) using a trial solution of the form:

$$\rho(x, t) = \sqrt{\frac{A(t)}{\pi}} \exp(-A(t)x^2)$$

Solution:

$$\frac{\partial \rho}{\partial t} = (\frac{1}{2A(t)} - x^2)\dot{A}(t)\rho(x, t)$$

$$\frac{\partial \rho}{\partial x} = -2xA(t)\rho(x, t)$$

$$\frac{\partial^2 \rho}{\partial x^2} = (-2A(t) + 4A(t)^2x^2)\rho(x, t)$$

Substitute in the diffusion equation and we get:

$$\dot{A}(t) = -4DA(t)^2$$

Next we try a power law solution: $A(t) = at^{-n}$, with $a > 0$ and $b > 0$ because we want $A(t)$ to go to infinity as $t \longrightarrow 0^+$.

From this we get:

$$t^{n-1} = \frac{4Da}{n}$$

Since the right side is independent of t: $n-1=0 \Rightarrow n=1$.
Therefore we conclude:

$$A(t) = \frac{1}{4Dt}$$

(b) Add a reflective boundary at $x=0$, and place a drop at a distance a from the boundary. Solve for the density $\rho(x,t)$.

Solution:

The density without the reflective boundary is given by:

$$\rho(x,T) = \frac{1}{\sqrt{4\pi Dt}} e^{-(x-a)^2/4Dt}$$

The reflective boundary condition is a Neumann B.C., the solution has to satisfy: $\nabla\rho(0,T)=0$, there is no current passing through the boundary.

By using the method of images, we can consider a second solution centered at $x=-a$.

$$\rho(x,T) = \frac{1}{\sqrt{4\pi Dt}} e^{-(x-a)^2/4Dt} + \frac{C}{\sqrt{4\pi Dt}} e^{-(x+a)^2/4Dt}$$

Differentiating and evaluating at the boundary $x=0$ we get that for the B.C. to be satisfied: $C=1$. The final solution is:

$$\rho(x,T) = \frac{1}{\sqrt{4\pi Dt}} e^{-(x-a)^2/4Dt} + \frac{1}{\sqrt{4\pi Dt}} e^{-(x+a)^2/4Dt}$$