Q Consider matter which initially has a uniform temperature T_0 and a two dimensional density profile $\rho(r, t=0) = \rho_0 \, e^{-r^2/2R_0^2}$,

and expands according to an equation of state $P = \rho T$ according to ideal hydrodynamics.

(a) Show that if the subsequent evolution of the density and temperature are parameterized by $\rho(r,t) = \rho_0 \dfrac{R_0^2}{R^2(t)} e^{-r^2/2R(t)^2}$, $T(r,t) = T(t)$.

then, the entropy will be conserved if $R^2(t)\,T(t) = R_0^2 T_0$.

Sol^n: The entropy per particle, $\sigma = S/N$, can be calculated by considering a single particle in area (A/N) in the canonical ensemble:

$$\sigma = \frac{S}{N} = \frac{\beta E + \ln Z_c}{N} = 1 + \ln\left[\frac{(A/N)}{(2\pi\hbar)^2}\int d^2p\, e^{-\beta \epsilon_p}\right] = 2 + \ln\left[\frac{1}{\rho}\left(\frac{mT}{2\pi\hbar^2}\right)\right]$$

The total entropy is then,

$$S(t) = \int d^2r\, \rho(r)\sigma(r) = \text{const} + \int d^2r\, \rho(r) \ln\left[T/\rho\right]$$

$$= \text{const} + \ln T \int d^2r\, \rho(r) - \int d^2r\, \rho(r) \ln \rho(r)$$

$$= \text{const} + \underbrace{2\pi\rho_0 R_0^2}_{\text{define this to be } N} \ln T - \int (2\pi r\, dr) \frac{\rho_0 R_0^2}{R^2} e^{-\frac{r^2}{2R^2}}\left[\ln\left(\frac{\rho_0 R_0^2}{R^2}\right) - \frac{r^2}{2R^2}\right]$$

$$= \text{const} + N\ln T - N\ln\left(\frac{\rho_0 R_0^2}{R^2}\right) + \frac{2\pi\rho_0 R_0^2}{2R^4}\underbrace{\int dr\, r^3 e^{-r^2/2R^2}}_{\propto R^4}$$

$$\Rightarrow S \simeq \text{const} + N\ln\left(TR^2\right).$$

Thus entropy will be conserved if $TR^2 = \text{const}$, ie $R^2(t)\,T(t) = R_0^2 T_0$. □.

[Fun Facts: In n-dim, we have $\rho = \rho_0 \dfrac{R_0^n}{R^n(t)} e^{-r^2/2R^2(t)}$, & we have

$\sigma = \text{const} + \ln\left[\dfrac{T^{n/2}}{\rho}\right]$ & $S(t) = \text{const} + \int d^n r\, \rho(r)\left[\ln T^{n/2} - \ln\rho\right]$

$$\Rightarrow S(t) = \text{const} + N\ln T^{n/2} - N\ln\left(\frac{\rho_0 R_0^n}{R^n(t)}\right) + \frac{k}{R^{n+2}}\underbrace{\int d\Omega\, dr\, r^{n+1} e^{-r^2/2R^2(t)}}_{\propto R^{n+2}}$$

$$\Rightarrow S(t) = \text{const} + N\ln\left(T^{n/2}R^n(t)\right) \Rightarrow R^2(t)\,T(t) = R_0^2 T_0 \;]$$

(b) Assuming that the velocity profile is linear, $v(r,t) = A(t)\, r$ find $A(t)$ and $R(t)$ that satisfy the hydrodynamics equation of motion and current conservation

Hydrodynamics eqn's:

$$\frac{Dv}{Dt} = -\frac{1}{m\rho}\nabla P$$

$$\frac{D\epsilon}{Dt} = -(P+\epsilon)\nabla\cdot v$$

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot v$$

Soln: Current conservation eqn leads to, $\dfrac{D\rho}{Dt} = -\rho\nabla\cdot v$

$$\Rightarrow \frac{\partial\rho}{\partial t} + v\frac{\partial\rho}{\partial r} + \rho\nabla\cdot v = 0.$$

$$\Rightarrow \rho\left[-\frac{2\dot{R}}{R} + \frac{r^2}{R^3}\dot{R} - Ar\frac{r}{R^2} + 2A\right] = 0.$$

which can be satisfied for all r if $\boxed{\dot{R}(t) = A(t)R(t)}$ ... (i)

Next we consider the equations of motion, ... [Note that this is 3 eqn's (or 2 here) (one for each v_i)]

$$\frac{Dv}{Dt} = -\frac{1}{m\rho}\nabla P \Rightarrow \frac{\partial v_i}{\partial t} + v\cdot\nabla v_i = -\frac{1}{m\rho}\nabla P$$

$$\Rightarrow \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = \frac{-1}{m\rho}\frac{\partial P}{\partial r} = \frac{-T}{m\rho}\frac{\partial\rho}{\partial r} \quad (\text{since } P = \rho T)$$

$$\Rightarrow \dot{A}r + A^2 r = \frac{Tr}{mR^2} \quad \rightarrow (\text{combining 2 eqn's})$$

This can be satisfied if, $\dot{A}(t) + A^2(t) = \dfrac{T(t)}{mR^2(t)}$

Using $R^2T = R_0^2T_0$, we get

$$\dot{A}(t) + A^2(t) = \frac{T_0 R_0^2}{m R^4(t)} \quad \text{....(ii)}$$

& we have $\dot{R}(t) = A(t)R(t)$ from eq (i)

Using eq (i) to eliminate A & $\dot{A}$ in eqn (ii), we get

$$\left[\dot{A} = \frac{\ddot{R} - A\dot{R}}{R} = \frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} \qquad A = \frac{\dot{R}}{R}\right]$$

$$m\ddot{R}(t) = \frac{R_0^2 T_0}{R^3(t)}$$

This can be solved by noting that the term on the right looks like a force affecting a particle of mass m at position R where the potential is $V(R) = R_0^2T_0/2R^2$. Given the potential, one can solve for the trajectory with $t = \displaystyle\int_{R_0}^{R}\frac{dx}{v} = \int_{R_0}^{R}\frac{dx}{\sqrt{2(E-V(x))/m}}$

$$\Rightarrow R(t)^2 = R_0^2 + \left(\frac{T_0}{m}\right)t^2; \quad A(t) = \left(\frac{T_0}{m}\right)\frac{t}{R^2(t)}; \quad T = \frac{T_0R_0^2}{R^2(t)}.$$