Homework Solutions

PHY 321 - Classical Mechanics I

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Contents

1 Chapter 1 Solutions

1. All physicists must become comfortable with thinking of oscillatory and wave mechanics in terms of expressions that include the form $e^{i\omega t}$.

   (a)

$$\begin{aligned} \cos\omega t &= 1 - \frac{1}{2}(\omega t)^2 + \frac{1}{4!}(\omega t)^4 + \frac{1}{6!}(\omega t)^6 \cdots \\ \sin\omega t &= (\omega t) - \frac{1}{3!}(\omega t)^3 + \frac{1}{5!}(\omega t)^5 \cdots \end{aligned}$$

   (b)

$$\begin{aligned} e^{i\omega t} &= 1 + i\omega t + \frac{1}{2}(i\omega t)^2 + \frac{1}{3}(i\omega t)^3 + \frac{1}{4!}(i\omega t)^4 \cdots \\ &= 1 + i^2\frac{1}{2}(\omega t)^2 + i^4\frac{1}{4!}(\omega t)^4 + i^6\frac{1}{6!}(\omega t)^6 \cdots + i\omega t + i^3\frac{1}{3!}(\omega t)^3 + i^5\frac{1}{5!}(\omega t)^5 \cdots \end{aligned}$$

   (c) Use fact that $i^2 = -1$ and $i^4 = 1$ then add expressions above for even and odd terms of expansions to see $e^{i\omega t} = \cos\omega t + i\sin\omega t$.

   (d)

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 \quad (1.1)$$

$$\ln(-1) = i\pi. \quad (1.2)$$

2. Find the angle between the vectors $\vec{b} = (1, 2, 4)$ and $\vec{c} = (4, 2, 1)$ by evaluating their scalar product.

$$\vec{b}\cdot\vec{c} = 1 + 4 + 4 = 12, \quad |\vec{b}| = \sqrt{1+4+16} = \sqrt{21}, \quad |\vec{c}| = \sqrt{16+4+1} = \sqrt{21} \quad (1.3)$$

$$\cos\theta_{bc} = \frac{\vec{b}\cdot\vec{c}}{|\vec{b}|\,|\vec{c}|} = \frac{12}{21}, \quad (1.4)$$

$$\theta_{bc} = \cos^{-1}(4/7). \quad (1.5)$$

3. Use the product rule to show that

$$\frac{d}{dt}(\vec{r}\cdot\vec{s}) = \frac{d\vec{r}}{dt}\cdot\vec{s} + \vec{r}\cdot\frac{d\vec{s}}{dt}.$$

   Solution

$$\frac{d}{dt}\sum_i r_i s_i = \sum_i \frac{dr_i}{dt}s_i + r_i\frac{ds_i}{dt} \quad (1.6)$$

$$= \frac{d\vec{r}}{dt}\cdot\vec{s} + \vec{r}\cdot\frac{d\vec{s}}{dt}. \quad (1.7)$$

4. **Multiply the rotation matrix in Example 1 by its transpose to show that the matrix is unitary, i.e. you get the unit matrix.**

   Solution:

$$\begin{aligned}
&\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
= &\begin{pmatrix} \cos^2\phi + \sin^2\phi & -\cos\phi\sin\phi + \cos\sin\phi & 0 \\ -\cos\phi\sin\phi + \cos\sin\phi & \cos^2\phi + \sin^2\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
= &\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

5. **Find the matrix for rotating a coordinate system by 90 degrees about the x axis.**

   Solution:

$$\begin{aligned}
U &= \begin{pmatrix} \hat{e}'_1\cdot\hat{e}_1 & \hat{e}'_1\cdot\hat{e}_2 & \hat{e}'_1\cdot\hat{e}_3 \\ \hat{e}'_2\cdot\hat{e}_1 & \hat{e}'_2\cdot\hat{e}_2 & \hat{e}'_2\cdot\hat{e}_3 \\ \hat{e}'_3\cdot\hat{e}_1 & \hat{e}'_3\cdot\hat{e}_2 & \hat{e}'_3\cdot\hat{e}_3 \end{pmatrix}, \qquad U_{ij} = \cos\theta_{ij} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}
\end{aligned}$$

   The signs of the off-diagonal term would switch depending on whether the coordinate system changes or the object.

6. **Consider a parity transformation which reflects about the $x = 0$ plane. Find the matrix that performs the transformation. Find the matrix that performs the inverse transformation.**

   Solution:
   The matrix needs to flip the x components of any vector while leaving the y and z components unchanged.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

7. Show that the scalar product of two vectors is unchanged if both undergo the same rotation. Use the fact that the rotation matrix is unitary, $U_{ij} = U^{-1}_{ji}$.
   Solution:

$$
\begin{aligned}
\vec{x}\cdot\vec{y} &= x_i y_y \\
\vec{x}'\cdot\vec{y}' &= U_{ij}x_j U_{ik} y_k \\
&= x_j U^t_{ji} U_{ik} y_k \\
&= x_j U^{-1}_{ji} U_{ik} y_k \\
&= x_j \delta_{jk} y_k = x_j y_j . \checkmark
\end{aligned}
$$

8. Show that the product of two unitary matrices is a unitary matrix.
   Solution:

$$
\begin{aligned}
(UV)^t UV &=? \quad \mathbb{I} \\
(UV)^t_{ij} &= \quad U_{jk} V_{ki} \\
U_{jk} V_{ki} U_{jm} V_{mn} &=? \delta_{in} \\
&= \quad V^t_{ik} U^t_{kj} U_{jm} V_{mn} \\
&= \quad V^t_{ik} V_{kb} = \delta_{in} . \checkmark
\end{aligned}
$$

9. **Show that**

$$
\sum_k \epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.
$$

   Solution:
   For any k in the sum, because $i \neq j$ and $l \neq m$, and because there are only two indices available because none can be equal to k, the only terms possible are $\delta_{ij}\delta jm$ and $\delta_{im}\delta_{jl}$. One then multiplies the matrices out for $i = 1, j = 2, l = 1, m = 2$, $i = 2, j = 3, l = 2, m = 3$, and $i = 1, j = 3, l = 1, m = 3$. There are also solutions with $i \leftarrow j$ and $l \leftrightarrow m$, but if you show these four, you know the ones with the indices switched will just switch the overall sign due to the antisymmetry of ϵ. To show the first case $(i = 1, j = 2, l = 1, m = 2)$,

$$
\sum_k \epsilon_{12k}\epsilon_{12k} = 1, \ \delta_{11}\delta 22 - \delta_{12}\delta_{21} = 1.
$$

   The others are similar.

10. Find min or max of $z = 3x^2 - 4y^2 + 12xy - 6x + 24$ Solution:

$$
\begin{aligned}
\partial_x z &= 6x + 12y - 6 = 0, && (1.8)\\
\partial_y z &= -8y + 12x = 0 && (1.9)\\
-32y + 12 &= 0, \ \ y = 3/8, && (1.10)\\
x &= 1/4. && (1.11)
\end{aligned}
$$

   To find whether max or min,

$$
\begin{aligned}
z(1/4, 3/8) &= -6.375 && (1.12)\\
z(0,0) &= -6. && (1.13)
\end{aligned}
$$

Suggests minimum. However, if you look more carefully, you will find this is a saddle point (Eigenvalues of $\partial_i\partial_j z$ matrix are mix of positive and negative. – You won't be expected to do this). You can also try a few sample points and see that some values are higher than the xy point found above and some are lower.

11. **Consider a cubic volume $V = L^3$ defined by $0 < x < L$, $0 < y < L$ and $0 < z < L$. Consider a vector $\vec{A}$ that depends arbitrarily on x, y, z. Show how Guass's law,**

$$\int_V dv \nabla \cdot \vec{A} = \int_S d\vec{S} \cdot \vec{A},$$

is satisfied by direct integration.

Solution:

$$\begin{aligned}
\int_V dv \nabla \cdot \vec{A} &= \int_0^{L_x} dx \int_0^{L_y} dy \int_0^{L_z} dz \, (\partial_x A_x + \partial_y A_y + \partial_z L_z) \\
&= \int_0^{L_y} dy \int_0^{L_z} dz \, [A_x(L_x, y, z) - A_x(0, y, z)] \\
&+ \int_0^{L_x} dx \int_0^{L_z} dz \, [A_y(x, L_y, z) - A_y(x, 0, z)] \\
&+ \int_0^{L_x} dx \int_0^{L_z} dy \, [A_z(x, y, L_z) - A_z(x, y, 0)] \\
&= \int_S d\vec{S} \cdot \vec{A}.
\end{aligned}$$

12. A real $n-$dimensional symmetric matrix λ can always be diagonalized by a unitary transformation, i.e. there exists some unitary matrix U such that,

$$U_{ij} \lambda_{jk} U^{-1}_{km} = \tilde{\lambda}_{im} = \begin{pmatrix} \tilde{\lambda}_{11} & 0 & \cdots & 0 \\ 0 & \tilde{\lambda}_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \tilde{\lambda}_{nn} \end{pmatrix}. \tag{1.14}$$

The values $\tilde{\lambda}_{ii}$ are referred to as eigenvalues. The set of n eigenvalues are unique, but their ordering is not – there exists a unitary transformation that permutes the indices.

Consider a function $f(x_1, \cdots, x_n)$ that has the property,

$$\partial_i f(\vec{x})|_{\vec{x}=0} = 0, \tag{1.15}$$

for all i. Show that if this function is a minimum, and not a maximum or an inflection point, that the n eigenvalues of the matrix

$$\lambda_{ij} \equiv \partial_i \partial_j f(\vec{x})|_{\vec{x}=0}, \tag{1.16}$$

must be positive.

Solution:

You can always calculate δf in the new coordinate system, and because f is a scalar, you can write

$$\begin{aligned}
\delta f &= \frac{1}{2}\delta r_i \delta r_j \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j} f(\vec{r}) \\
&= \frac{1}{2}\delta r'_i \delta r'_j \frac{\partial}{\partial r'_i}\frac{\partial}{\partial r'_j} f(\vec{r}') \\
&= \frac{1}{2}\sum_i \delta r'_i \delta r'_i \tilde{\lambda}_{ii}.
\end{aligned}$$

4 Because you can adjust the components of $\delta r'$ individually, then each component of $\tilde{\lambda}$, i.e. each eigenvalue of λ, must be positive.

13. (a) For the unitary matrix U that diagonalizes λ as shown in the previous problem. Show that each row of the unitary matrix represents an orthogonal unit vector by using the definition of a unitary matrix.

    (b) Show that the vector

$$x_i^{(k)} \equiv U_{ki} = (U_{k1}, U_{k2}, \cdots, U_{kn}), \tag{1.17}$$

has the property that

$$\lambda_{ij} x_j^{(k)} = \tilde{\lambda}_{kk} x_i^{(k)}. \tag{1.18}$$

These vectors are known as eigenvectors, as they have the property that when multiplied by λ the resulting vector is proportional (same direction) as the original vector. Because one can transform to a basis, using U, where λ is diagonalized, in the new basis the eigenvectors are simply the unit vectors.

Solution:

$$U^t_{ji} U_{ik} = \delta_{ik} \tag{1.19}$$

$$x_i^{(j)} \equiv U^t_{ji} = U_{ij}, \tag{1.20}$$

$$x_i^{(j)} x_i^{(k)} = \delta_{ik}. \checkmark \tag{1.21}$$

14. Which of these equations might be valid? Base your response solely on whether whether both sides of the equation are consistent dimensionally (center column) or rotationally (right-hand column). Here, $F, q, v, B, L, r, t, p, E, a$ refer to force, charge, velocity, magnetic field, angular momentum, position, time, momentum and energy, respectively.

expression	dimensionally	rotationally
$F_i = q\epsilon_{ijk} v_j B_k$ (same as $\vec{F} = q\vec{v} \times \vec{B}$)	✓	✓
$\vec{F} = \vec{r} \cdot \vec{L}/r^2$		
$r_i = v_i t + \frac{t^2}{2}\epsilon_{ijk} a_k$	✓	
$E = \vec{r} \cdot (\vec{p} \times \vec{L})$		✓

2 Chapter 2 Solutions

1. Consider a bicyclist with air resistance proportional to v^2 and rolling resistance proportional to v, so that
$$\frac{dv}{dt} = -Bv^2 - Cv.$$
If the cyclist has initial velocity v_0 and is coasting on a flat course, a) find her velocity as a function of time, and b) find her position as a function of time.

   Solution:

$$\begin{aligned}
t &= -\int_{v_0}^{v} dv' \, \frac{1}{Bv'^2 + Cv'} \\
Bt &= -\frac{1}{\beta}\int_{v_0}^{v} dv' \left(\frac{1}{v'} - \frac{1}{v'+\beta}\right), \quad \beta \equiv C/B \\
&= \frac{-1}{\beta}\ln\left(\frac{v(v_0+\beta)}{v_0(v+\beta)}\right), \\
v &= \frac{v_0 e^{-Ct}}{1+(Bv_0/C)(1-e^{-Ct})}, \\
x &= \int dt\, v(t) = \frac{-1}{C}\int du\, \frac{v_0}{1+Bv_0(1-u)/C}, \quad u \equiv e^{-Ct} \\
&= \frac{1}{B}\ln\left(1+Bv_0(1-u)/c\right) \\
&= \frac{1}{B}\ln\left[1+Bv_0(1-e^{-Ct})/C\right].
\end{aligned}$$

2. For Eq. (36) show that in the limit where $\gamma \to 0$ one finds $t = 2v_{0y}/g$.

   Solution:
   Eq. (36) is
$$0 = -\frac{gt}{\gamma} + \frac{v_{0y}+g/\gamma}{\gamma}\left(1-e^{-\gamma t}\right).$$
Expand the exponential to 2nd order in a Taylor expansion,
$$\begin{aligned}
0 &= -\frac{gt}{\gamma} + \frac{v_{0y}+g/\gamma}{\gamma}\left(\gamma t - \frac{1}{2}\gamma^2 t^2 - \cdots\right) \\
0 &= v_{0y}t - \frac{1}{2}gt^2, \\
t &= 2v_{0y}/g.
\end{aligned}$$

3. The motion of a charged particle in an electromagnetic field can be obtained from the **Lorentz equation**. If the electric field vector is **E** and the magnetic field is **B**, the force on a particle of mass m that carries a charge q and has a velocity **v**
$$\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}$$

where we assume $v << c$(speed of light).

(a) If there is no electric field and if the particle enters the magnetic field in a direction perpendicular to the lines of magnetic flux, show that the trajectory is a circle with radius

$$r = \frac{mv}{qB} = \frac{v}{\omega_c},$$

where $\omega_c \equiv qB/m$ is the *cyclotron frequency*.

(b) Choose the z-axis to lie in the direction of B and let the plane containing $\mathbf{E}$ and $\mathbf{B}$ be the $yz-$plane. Thus

$$\mathrm{B} = B\hat{k}, \quad \mathrm{E} = E_y\hat{j} + E_z\hat{k}.$$

Show that the z component of the motion is given by

$$z(t) = z_0 + \dot{z}_0 t + \frac{qE_z}{2m}t^2,$$

where

$$z(0) \equiv z_0 \ \text{ and } \ \dot{z}_0 \equiv \dot{z}_0.$$

(c) Continue the calculation and obtain expressions for $\dot{x}(t)$ and $\dot{y}(t)$. Show that the time averages of these velocity components are

$$\langle \dot{x} \rangle = \frac{E_y}{B}, \quad \langle \dot{y} \rangle = 0.$$

(Show that the motion is periodic and then average over one complete period.)

(d) Integrate the velocity equations found in (c) and show (with the initial conditions $x(0) = -A/\omega_c,\ \dot{x}(0) = E_y/B,\ y(0) = 0,\ \dot{y}(0) = A$ that

$$x(t) = \frac{-A}{\omega_c}\cos\omega_c t + \frac{E_y}{B}t, \quad y(t) = \frac{A}{\omega_c}\sin\omega_c t.$$

These are the parametric equations of a trochoid. Sketch the projections of the trajectory on the xy-plane for the cases (i) $A > |E_y/B|$, (ii) $A < |E_y/B|$, and (iii) $A = |E_y/B|$. (The last case yields a cycloid.)

Solution:

(a) The solution found in Eq. (43)

$$\begin{aligned}
x - x_0 &= \frac{-A}{\omega_c}\cos(\omega_c t - \phi), \quad v_x = A\sin(\omega_c t - \phi), \\
y - y_0 &= \frac{A}{\omega_c}\sin(\omega_c t - \phi), \quad v_y = A\cos(\omega_c t - \phi), \\
\omega_c &\equiv \frac{qB}{m}.
\end{aligned}$$

The velocity is $v = A$ so the radius of the orbit is $r = A/\omega_c = mv/qB$.

(b) Since there are no magnetic forces in the z direction, the z motion is determined by

$$ma_z = qE_z,$$

which is constant acceleration $a_z = qE_z/m$, and for initial position z_0 and initial $v_z = v_{z0}$,

$$z = z_0 + v_{0z}t + \frac{1}{2}\frac{qE_z}{m}t^2.$$

(c) The equations of motion are

$$\begin{aligned} \dot{v}_y &= -\omega_c v_x + qE_y/m, \\ \dot{v}_x &= \omega_c v_y, \end{aligned}$$

Substituting,

$$\begin{aligned} \ddot{v}_y &= -\omega_c^2 v_y, \\ \ddot{v}_x &= -\omega_c^2 v_x + qE_y\omega_c/m, \\ \frac{d^2}{dt^2}(v_x - qE_y\omega_c/m) &= -\omega_c^2(v_x - E_y/B)). \end{aligned}$$

By inspection, $\langle v_y \rangle = 0$ and $\langle v_x \rangle = E_y/B$.

(d) From the equations above, one can see that for the initial conditions $\dot{y}(0) = A$, $\dot{x}(0) = 0$,

$$\begin{aligned} v_y(t) &= A\cos\omega_c t, \\ v_x(t) &= -A\sin\omega_c t + E_y/B. \end{aligned}$$

Integrating over time, and assuming $y(0)$=0 and $x(0) = A/\omega_c$,

$$y(t) = (A/\omega)\sin\omega_c t, \quad x(t) = -(A/\omega)\cos\omega_c t + \frac{E_y t}{B}.$$

4. A particle of mass m has velocity $v = \alpha/x$, where x is its displacement. Find the force $F(x)$ responsible for the motion.

Solution:

$$\begin{aligned} U(x) + \frac{1}{2}mv^2 &= \text{constant}, \\ U(x) &= -\text{constant} - \frac{m\alpha^2}{2x^2}, \\ F(x) &= -\partial_x U(x) = -\frac{m\alpha^2}{x^3}. \end{aligned}$$

5. A particle is under the influence of a force $F = -kx + kx^3/\alpha^2$, where k and α are constants and k is positive. Determine $U(x)$ and discuss the motion. What happens when $E = (1/4)k\alpha^2$?

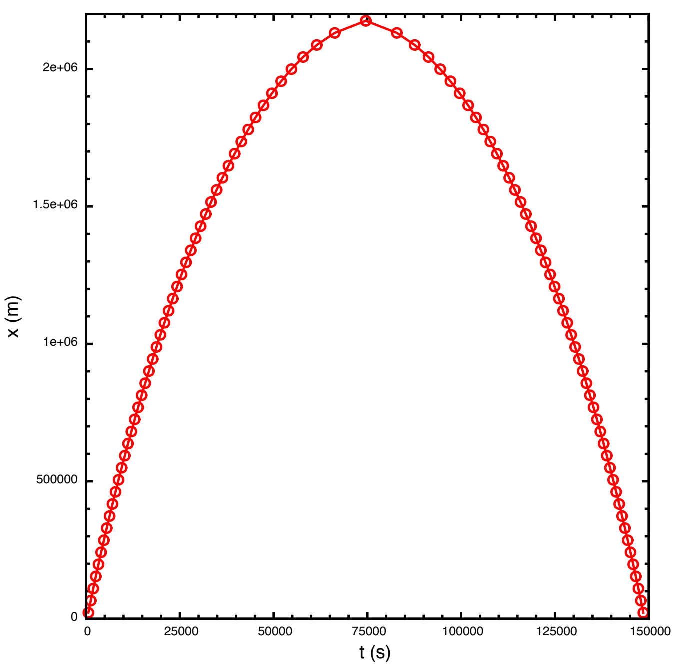
Solution:

$$U(x) = \int dx F(x) = \frac{1}{2}kx^2 - \frac{1}{4\alpha^2}kx^4.$$

This potential has a well at $x = 0$ then rises until $x = \alpha$ before falling off again. The maximum potential, $U(x = \pm\alpha) = k\alpha^2/4$. If the energy is higher, the particle cannot be contained in the well.

6. Using Eq. (**??**) find the position as a function of a time by numerical integration for the case where a particle of mass m moves under the potential $U(x) = U_0\sqrt{x/L}$ and for an initial velocity v_0. Use the following values: $m = 2.5kg$, $v_0 = 75.0$, $L = 10m$, U_0=15 J. Solve for time until the particle returns to the origin. Make a graph of x vs t for the entire trajectory from your computer output. Turn in the graph and a printout of your program.

Solution:




```
int main(){
  double t=0.0,x,U,KE,v,U0=15,L=10.0,v_0=75.0,m=2.5;
  double E=0.5*m*v_0*v_0;
  double xmax=L*pow(E/U0,2);
  printf("xmax=%g\n",xmax);
  double dx=xmax/50.0;
  for(x=0.5*dx;x<xmax;x+=dx){
    U=U0*sqrt(x/L);
    KE=E-U;
    v=sqrt(2.0*KE/m);
    t+=dx/v;
    printf("%8.1lf %10.1lf %g\n",t,x,v);
```

```
  }
  printf("tmax=%g\n",t);
  return 0;
}
```

7. Prove that the work on the center of mass during a small time interval δt, which is defined by

$$\Delta W_{\rm cm} \equiv \vec{F}_{\rm tot} \cdot \delta\vec{r}_{\rm cm},$$

is equal to the change of the kinetic energy of the center of mass,

$$T_{\rm cm} \equiv \frac{1}{2} M_{\rm tot} v_{\rm cm}^2, \qquad \Delta T_{\rm cm} = M_{\rm tot} \vec{v}_{\rm cm} \cdot \delta\vec{v}_{\rm cm}.$$

Use the following definitions:

$$\begin{aligned}
\vec{F}_{\rm tot} &= \sum_i \vec{F}_i, \\
M_{\rm tot} &= \sum_i m_i, \\
\vec{r}_{cm} &= \frac{1}{M_{\rm tot}} \sum_i m_i \vec{r}_i, \\
\vec{v}_{cm} &= \frac{1}{M_{\rm tot}} \sum_i m_i \vec{v}_i.
\end{aligned}$$

Solution:

$$\begin{aligned}
\sum_i \vec{F}_i \cdot \frac{\sum_j m_j \delta\vec{r}_j}{\sum_k m_k} &=? \ (\sum_k m_k) \frac{\sum_{ij} m_i \vec{v}_i \cdot m_j \delta\vec{v}_j}{(\sum_k m_k)^2} \\
\sum_{ij} m_i \frac{\delta\vec{v}_i}{\delta t} \cdot m_j \delta\vec{r}_j &=? \ \sum_{ij} m_i \vec{v}_i \cdot m_j \delta\vec{v}_j
\end{aligned}$$

Because $\delta\vec{r}_i/\delta t = \vec{v}_i$, both sides are equal.

8. Consider a rocket with initial mass M_0 at rest in deep space. It fires its engines which eject mass with an exhaust speed v_e relative to the rocket. The rocket loses mass at a constant rate $\alpha = dM/dt$. Find the speed of the rocket as a function of time.

Solution:

The rocket's speed is

$$v = -v_e \ln\left(\frac{M(t)}{M_0}\right).$$

Here $M(t) = M - \alpha t$, so

$$v(t) = -v_e \ln\left(\frac{M_0 - \alpha t}{M_0}\right).$$

9. Imagine that a rocket can be built so that the best percentage of fuel to overall mass is 0.9. Explain the advantage of having stages.

Solution:
The change in velocity during the final stage is $-v_e \ln(1-f)$, where f is the fraction of the mass that is fuel. Thus, any acceleration during earlier stages adds to the overall speed. If there were only one large stage, the final answer would be set by f and would be independent of the size of the rocket.

10. Ted and his iceboat have a combined mass of 200 kg. Ted's boat slides without friction on the top of a frozen lake. Ted's boat has a winch and he wishes to wind up a long heavy rope of mass 300 kg and length 100 m that is laid out in a straight line on the ice. Ted's boat starts at rest at one end of the rope, then brings the rope on board the ice boat at a constant rate of 0.25 m/s. After 400 meters the rope is all aboard the iceboat.

    (a) Before Ted turns on the winch, what is the position of the center of mass relative to the boat?

    (b) Immediately after Ted starts the winch, what is his speed?

    (c) Immediately after the rope is entirely on the boat, what is Ted's speed?

    (d) Immediately after the rope is entirely on board, what is Ted's displacement relative to his original position?

    (e) Immediately after the rope is entirely on board, where is the center of mass compared to Ted's original position?

    (f) Find Ted's velocity as a function of time.

Solution:
a) $R = (200 \cdot 0 + 300 \cdot 50)/500 = 30$ m.

b) $v_t - v_r = 0.25$ m/s, $200 \cdot v_t + 300 \cdot v_r = 0$. Solve 2 eq.s 2 unk.s and get $v_t = 0.15$ m/s.

c) Zero, by momentum conservation

d) 30 m by conservation of center-of-mass

e) same as (d)

f) Since the time-dependent masses are

$$m_t = 200 + 0.25t\frac{300}{100},\ m_r = 300 - 0.25t\frac{300}{100}.$$

the equations to solve are

$$\begin{aligned} v_t - v_r &= 0.25 \\ (200 + 0.75t)v_t + (300 - 0.75t)v_r &= 0. \end{aligned}$$

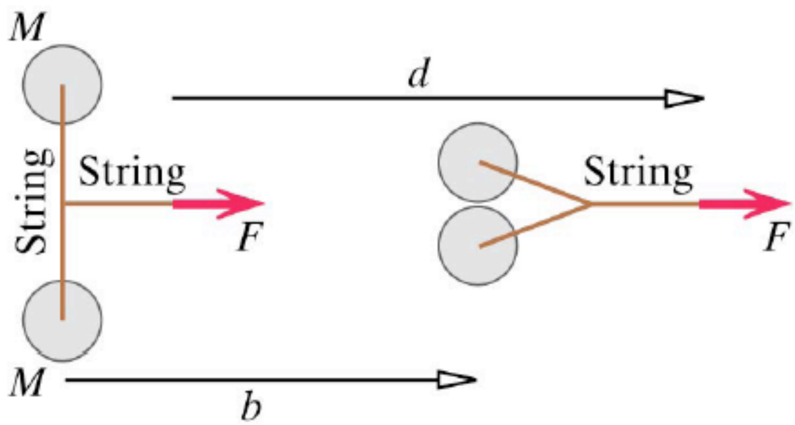
Solving for Ted's velocity,

$$v_t(t) = 0.25\frac{300 - 0.75t}{500}.$$

Integrate to get Ted's position

$$x = \int_0^t dt' \, v_t(t') = 0.15t - \frac{3}{16000}t^2.$$

11. Two disks are initially at rest, each of mass M, connected by a string between their centers. The disks slide on low-friction ice as the center of the string is pulled by a string with a constant force F through a distance d. The disks collide and stick together, having moved a distance b horizontally. Determine the final speed of the disks just after they collide.



Solution:
The horizontal force acting on the upper disk is $F/2$, the "work" along the x direction is

$$Fb/2 = \frac{1}{2}mv_x^2.$$

The x component of the velocity is unchanged by the sudden "sticking" of the two disks, so,

$$v_x = \sqrt{Fb/m}.$$

Or, Consider the center-of-mass of the two-disk system. This moves according to the net force F a distance b. So, once again $Fx = (1/2)(2m)v^2$, and $v = sqrt Fb/m$.

12. Santa Claus is skating on the magic ice near the north pole, which is frictionless. A massless rope sticks out from the pole horizontally along a straight line. The rope's original length is L_0. Santa approaches the rope moving perpendicular to the direction of the rope and grabs the end of the rope. The rope then winds around the thin pole until Santa is half the original distance, $L_0/2$, from the pole. If Santa's original speed was v_0, what is his new speed?

Solution:

The string doesn't absorb energy, so the kinetic energy is unchanged and the final speed is v_0. Also if you consider a small pole of radius b, the torque is $Tb = m(v^2/r)b$.

$$\frac{d}{dt}(mvr) = m\dot{v}r + mv\dot{r} = Tb = mv^2b/r.$$

The rate at which the rope shortens is $\dot{r} = 2\pi b \frac{2\pi r}{v} = bv/r$. Plugging this in above,

$$m\dot{v}r + mv(bv/r) = mv^2 b/r, \quad m\dot{v}r = 0.$$

Thus v is constant.

3 Chapter 3 Solutions

1. A floating body of uniform cross-sectional area A and of mass density ρ and at equilibrium displaces a volume V. Show that the period of small oscillations about the equilibrium position is given by
$$\tau = 2\pi\sqrt{V/gA}$$

   Solution:
   The buoyant force is $\rho_{\text{fluid}}Agy$, where y is the height of the waterline from the bottom. The effective spring constant is $k = \rho_{\text{fluid}}Ag$ and the mass is ρV_{obj}, where V_{obj} is the volume of the object. The frequency is
$$\omega_0 = \sqrt{k/m} = \sqrt{\frac{\rho_{\text{fluid}}Ag}{\rho V_{\text{obj}}}}.$$
   From Archimedes principle $\rho V_{\text{obj}} = \rho_{\text{fluid}}V$. Thus
$$\omega_0 = \sqrt{\frac{Ag}{V}}, \quad \tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{V}{gA}}.$$

2. Show that the critically damped solution, Eq. (**??**), is indeed the solution to the differential equation.

   Solution:
   Use the fact that $\omega_0 = \beta$.
$$\begin{aligned} x &= Ae^{-\beta t} + Bte^{-\beta t} \\ \ddot{x} + 2\beta\dot{x} + \beta^2 x &= A\left\{\beta^2 - 2\beta^2 + \beta^2\right\} e^{-\beta t} + B\left\{(\beta^2 - 2\beta^2 + \beta^2)t + (-2\beta + 2\beta)\right\} \\ &= 0. \end{aligned}$$

3. Consider an over-damped harmonic oscillator with a mass of $m = 2$ kg, a damping factor $b = 20$ Ns/m, and a spring constant $k = 32$ N/m. If the initial position is $x = 0.125$ m, and if the initial velocity is -2.0 m/s, find and graph the motion as a function of time. Solve for the time at which the mass crosses the origin.

   Solution:
   The general solution is
$$x = A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}, \quad \beta_1 = \beta + \sqrt{\beta^2 - \omega_0^2}, \beta_2 = \beta - \sqrt{\beta^2 - \omega_0^2}.$$
   Here $\beta_1 = 8$ and $\beta_2 = 2$. The I.C. give
$$\begin{aligned} 0.125 &= A_1 + A_2, \\ -(\beta_1 A_1 + \beta_2 A_2) &= -2 \\ A_1 &= 7/24, \\ A_2 &= -1/6. \end{aligned}$$

The solution is

$$x = (7/24)e^{-8t} - (1/6)e^{-2t}.$$

This starts above the axis crosses the axis once, then bottoms out and approaches the axis from below. The point it crosses the axis is given by

$$(7/24)e^{-6t} = 1/6, \quad t = \frac{1}{6}\ln(7/4).$$

4. Consider a particle of mass m moving in a one-dimensional potential,

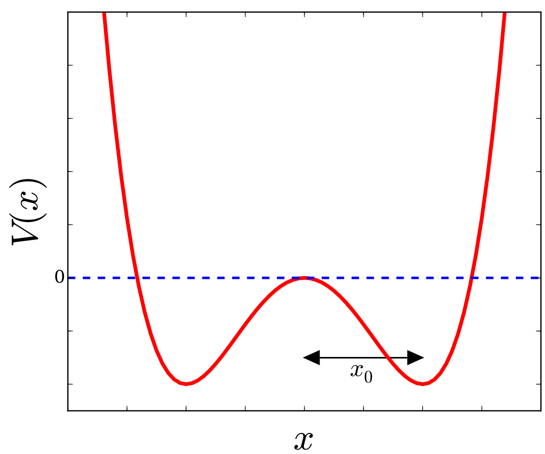
$$V(x) = -k\frac{x^2}{2} + \alpha\frac{x^4}{4}.$$

(a) What is the angular frequency for small vibrations about the minimum of the potential? What is the effective spring constant?

(b) If you add a small force $F = F_0 \cos(\omega t - \phi)$, and if the particle is initially at the minimum with zero initial velocity, find its position as a function of time.

(c) If there is a small drag force $-bv$, repeat (b).

Solution:
The potential looks like:



Solve for the position of the minimum,

$$\begin{aligned} \partial_x V(x) &= 0 \\ -kx + \alpha x^3 = 0, \quad x_{\rm min} &= \sqrt{k/\alpha}. \end{aligned}$$

The effective spring constant is the curvature at the minimum, i.e. the second derivative.

$$\begin{aligned} k_{\rm eff} &= \partial_x^2 V(x)\big|_{x_{\rm min}} \\ &= -k + 3\alpha x_{\rm min}^2 = 2k. \end{aligned}$$

The angular frequency for small vibrations is then

$$\omega_0 = \sqrt{2k/m}.$$

This is the same as the solution for a regular particle, but with an offset by $x_{\rm min}$. Take solution from notes with $\beta\neq 0$, then set $\beta=0$. Note that δ will equal zero.

$$x = x_{\rm min}+A_1\cos\omega_0 t+A_2\sin\omega_0 t+\frac{(F_0/m)\cos(\omega t-\phi)}{|\omega^2-\omega_0^2|}. \tag{3.1}$$

To solve for A_1 and A_2 consider the initial conditions,

$$\begin{aligned} 0 &= A_1+\frac{(F_0/m)\cos\phi}{|\omega^2-\omega_0^2|}\\ 0 &= \omega_0 A_2-\frac{(F_0\omega/m)\sin\phi}{|\omega^2-\omega_0^2|}. \end{aligned}$$

This coefficients are,

$$A_1=-\frac{(F_0/m)\cos\phi}{|\omega^2-\omega_0^2|},\quad A_2=\frac{(F_0\omega/m)\sin\phi}{\omega_0|\omega^2-\omega_0^2|}.$$

If we add a damping force, and define $\beta=b/2m$, the general solutions are

$$x = x_{\rm min}+A_1e^{-\beta t}\left[\cos\omega' t+A_2\sin\omega' t\right]+\frac{(F_0/m)\cos(\omega t-\phi-\delta)}{D}, \tag{3.2}$$

$$D \equiv \sqrt{(\omega_0-\omega^2)^2+4\beta^2\omega^2},\ \ \omega'\equiv\sqrt{\omega_0^2-\beta^2},\quad \delta\equiv\arctan(2\beta\omega/(\omega_0^2-\omega^2)) \tag{3.3}$$

The IC then constrain A_1 and A_2,

$$\begin{aligned} 0 &= A_1+\frac{(F_0/m)\cos(\phi+\delta)}{D}\\ 0 &= \omega' A_2-\frac{(F_0\omega/m)\sin(\phi+\delta)}{D},\\ A_1 &= -\frac{(F_0/m)\cos(\phi+\delta)}{D},\\ A_2 &= \frac{(F_0\omega/m)\sin(\phi+\delta)}{\omega' D}. \end{aligned}$$

5. Consider the periodic force, $F(t+\tau)=F(t)$,

$$F(t)=\left\{\begin{array}{ll} -A, & -\tau/2<t<0\\ +A, & 0<t><\tau/2\end{array}\right.$$

Find the coefficients f_n and g_n defined in Eq. (**??**).

Solution:
Since F is an odd function, $f_n=0$ and we have only the coefficients g_n from Eq. (**??**)

$$\begin{aligned} g_n &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2}F(t')\sin(2n\pi t'/\tau)dt'\\ &= \frac{4A}{\tau}\int_0^{\tau/2}\sin(2n\pi t'/\tau)dt'\\ &= \left\{\begin{array}{ll}\frac{4A}{n\pi}, & n=\text{odd}\\ 0, & n=\text{even}\end{array}\right. \end{aligned}$$

6. A "delta" function is a function that is zero everywhere except where the argument is zero. At this point the function is infinite so that the area under the curve is unity. The delta function obeys the relation

$$\int dt' f(t')\delta(t'-t_0) = f(t_0).$$

(a) Show that the following function

$$\left.\frac{1}{\pi}\frac{\Lambda}{\Lambda^2+x^2}\right|_{\Lambda\to 0} = \delta(x).$$

i.e. show that it is zero everywhere except the origin and that it integrates to unity.

(b) A step function, $\Theta(t)$, a.k.a. the "Theta" function or the Heaviside function, is zero for negative arguments and is unity for positive arguments. Show that

$$\frac{d}{dx}\Theta(x-x_0) = \delta(x-x_0).$$

(c) Using the definition of Fourier coefficients in Eq.s (**??**) and (**??**), show that

$$\delta(t-t_0) = -\frac{1}{\tau} + \frac{2}{\tau}\sum_{n=0}^{\infty}\cos(\omega_n(t-t_0)), \quad \omega_n = 2n\pi/\tau.$$

Solution: a) The function is clearly zero for $x \neq x_0$ and infinite for $x = x_0$. To check that it integrates to unity,

$$\begin{aligned}
\frac{1}{\pi}\int_{-\infty}^{\infty} dx\, \frac{\Lambda}{\Lambda^2+x^2} &= \ , \quad u \equiv x/\Lambda \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} du \frac{1}{1+u^2}, \quad u \equiv \tan\theta, du = \sec^2\theta d\theta, \\
&= \frac{1}{\pi}\int_{\tan^{-1}(-\infty)}^{\tan^{-1}(\infty)} d\theta = 1, \quad \text{because } \tan^{-1}(\infty) = \pi/2.
\end{aligned}$$

b) For all a and b

$$\int_a^b dx\, \frac{d}{dx}\Theta(x) = \Theta(b) - \Theta(a),$$

$$\int_a^b dx\, \delta(x) = \Theta(b) - \Theta(a),$$

so integrands must be equal.

c)

$$
\begin{aligned}
f_n &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} dt' \cos(n\omega t')\delta(t'-t_0) = \frac{2}{\tau}\cos(n\omega t_0),\\
g_n &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} dt' \sin(n\omega t')\delta(t'-t_0) = \frac{2}{\tau}\sin(n\omega t_0),\\
\delta(t-t_0) &= \frac{1}{\tau}+\frac{2}{\tau}\sum_{n>0}\{\cos(n\omega t_0)\cos(n\omega t)+\sin(n\omega t_0)\sin(n\omega t)\}\\
&= \frac{1}{\tau}+\frac{2}{\tau}\sum_{n>0}\cos(n\omega(t-t_0)),\\
&= -\frac{1}{\tau}+\frac{2}{\tau}\sum_{n\geq 0}\cos(n\omega(t-t_0)).
\end{aligned}
$$

7. Consider the complex function in the interval $-\tau/2 < t < \tau/2$,

$$
f(t) = -\frac{1}{\tau}+\frac{2}{\tau}\sum_{n=0}^{\infty} e^{in\omega(t-t_0)}, \quad \omega = 2\pi/\tau.
$$

(a) Using the fact that if one integrates over the interval, $-\tau/2 < t < \tau/2$, that $\int dte^{in\omega t} = 0$ for $n \neq 0$, show that

$$
\int dtf(t) = 1.
$$

(b) Using the fact that $\sum_n x^n = 1/(1-x)$, show that

$$
f(t) = -\frac{1}{\tau}+\frac{2/\tau}{1-e^{i\omega(t-t_0)}}.
$$

(c) From the expression in (b), show that the real part of $f(t)$ obeys

$$
\Re f(t) = 0, \quad \text{for } t \neq t_0
$$

This shows that $\Re f$ is a delta function and validates the result of the previous problem.

Solution:
(a) The integral for all $n \neq 0$ is zero, thus

$$
\begin{aligned}
\int_{-\tau/2}^{\tau/2} dt\, f(t) &= \int_{-\tau/2}^{\tau/2} dt \left\{\frac{-1}{\tau}+\frac{2}{\tau}\right\}\\
&= 1.
\end{aligned}
$$

(b) Let $x = e^{i\omega(t-t_0)}$,

$$
f(t) = \frac{-1}{\tau}+\frac{2}{\tau}\frac{1}{1-x} = \frac{-1}{\tau}+\frac{2}{\tau}\frac{1}{1-e^{i\omega(t-t_0)}}.
$$

(c)

$$\begin{aligned}
\Re f(t) &= \frac{-1}{\tau} + \frac{1}{\tau}\left\{\frac{1}{1-e^{i\omega(t-t_0)}} + \frac{1}{1-e^{-i\omega(t-t_0)}}\right\} \\
&= \frac{-1}{\tau} + \frac{1}{\tau}\left\{\frac{1-e^{-i\omega(t-t_0)}}{(1-e^{i\omega(t-t_0)})(1-e^{-i\omega(t-t_0)})} + \frac{1-e^{i\omega(t-t_0)}}{(1-e^{-i\omega(t-t_0)})(1-e^{i\omega(t-t_0)})}\right\} \\
&= \frac{-1}{\tau} + \frac{1}{\tau}\left\{\frac{2-e^{-i\omega(t-t_0)}-e^{i\omega(t-t_0)}}{2-e^{-i\omega(t-t_0)}-e^{i\omega(t-t_0)}}\right\} \\
&= -\frac{-1}{\tau} + \frac{1}{\tau} = 0.
\end{aligned}$$

8. A particle of mass m in an undamped harmonic oscillator with angular frequency ω_0 is at rest in the bottom of the well, when it experiences a force

$$F(t) = \begin{cases} 0, & t < 0 \\ G, & 0 < t < \tau \\ 0, & t > \tau \end{cases}$$

Find $x(t)$ for $t > \tau$.

Solution::
From Eq. (**??**),

$$\begin{aligned}
x(t) &= \frac{1}{m\omega'}\int_{-\infty}^{t} dt'\, F(t')e^{-\beta(t-t')}\sin[\omega'(t-t')] \\
&= \frac{G}{m\omega'}\int_0^{\tau} dt' e^{-\beta(t-t')}\sin[\omega'(t-t')] \\
&= \frac{G}{2im\omega'}\int_0^{\tau} dt'\left[e^{(-\beta+i\omega')(t-t')} - e^{(-\beta-i\omega')(t-t')}\right] \\
&= \frac{G}{2im\omega'}\left\{\frac{e^{(-\beta+i\omega')(t-\tau)} - e^{(-\beta+i\omega')t}}{(-\beta+i\omega')} - \frac{e^{(-\beta-i\omega')(t-\tau)} - e^{(-\beta-i\omega')t}}{(-\beta-i\omega')}\right\} \\
&= -\frac{G}{m\omega'}e^{-\beta t}\Im\left\{\frac{e^{i\phi}(e^{\beta\tau}e^{i\omega'(t-\tau)} - e^{i\omega' t})}{\sqrt{\beta^2+\omega'^2}}\right\}, \quad \phi \equiv -\arctan\omega'/\beta \\
&= \frac{-G}{m\omega'}e^{-\beta t}\left\{\frac{e^{\beta\tau}\sin(\omega'(t-\tau)-\phi) - \sin(\omega' t-\phi)}{\sqrt{\beta^2+\omega'^2}}\right\}
\end{aligned}$$

9. Consider a particle of mass m in a harmonic oscillator with angular frequency ω_0 and no damping. It experiences an external force,

$$F(t) = f_0\Theta(t)e^{-\gamma t}.$$

A "Theta" function is a step function, and is zero for negative arguments and unity for positive arguments.

(a) Find a particular solution, $x_p(t)$, assuming it is proportional to $e^{-\gamma t}$.

(b) For a particle initially at rest at the origin at $t = 0$, find $x(t)$ by adding in the homogenous solutions and matching the BC to determine the arbitrary constants.

(c) Use Eq. (**??**) to find $x(t)$. Check that you get the same result as (b).

Solution:

(a)

$$\begin{aligned} x_p(t) &= De^{-\gamma t}, \\ \left(\gamma^2 + \omega_0^2\right) De^{-\gamma t} &= (f_0/m)e^{-\gamma t}, \\ D &= \frac{f_0/m}{\gamma^2 + \omega_0^2}. \end{aligned}$$

(b)

$$x = A\cos\omega_0 t + B\sin\omega_0 t + De^{-\gamma t}, \tag{3.4}$$

$$0 = A + D, \quad A = -D = -\frac{f_0/m}{\gamma^2 + \omega_0^2}, \tag{3.5}$$

$$v = -\omega_0 A\sin\omega_0 t + \omega_0 B\cos\omega_0 t - \gamma De^{-\gamma t} \tag{3.6}$$

$$0 = \omega_0 B - \gamma D, \tag{3.7}$$

$$B = \frac{\gamma}{\omega_0} D. \tag{3.8}$$

(c)

$$x(t) = \frac{f_0}{m\omega_0}\int_0^t dt'\, e^{-\gamma t'}\sin\omega_0(t - t'), \tag{3.9}$$

$$= \frac{f_0}{2m\omega_0 i}\int_0^t dt'\, e^{-\gamma t'}\left(e^{i\omega_0(t-t')} - e^{-i\omega_0(t-t')}\right) \tag{3.10}$$

$$= \frac{f_0}{2m\omega_0 i}\left(\frac{e^{-\gamma t} - e^{i\omega_0 t}}{-\gamma - i\omega_0} - \frac{e^{-\gamma t} - e^{-i\omega_0 t}}{-\gamma + i\omega_0}\right) \tag{3.11}$$

$$= \frac{f_0}{2m\omega_0 i}\frac{(e^{-\gamma t} - e^{i\omega_0 t})(-\gamma + i\omega_0) - (e^{-\gamma t} - e^{-i\omega_0 t})(-\gamma - i\omega_0)}{\gamma^2 + \omega_0^2} \tag{3.12}$$

$$= \frac{f_0}{2m\omega_0 i(\gamma^2 + \omega_0^2)}\left\{\gamma e^{i\omega_0 t} - \gamma e^{-i\omega_0 t} + 2i\omega_0 e^{-\gamma t} - i\omega_0 e^{i\omega_0 t} + i\omega_0 e^{-i\omega_0 t}\right\} \tag{3.13}$$

$$= \frac{f_0}{m\omega_0(\gamma^2 + \omega_0^2)}\left\{\gamma\sin\omega_0 t - \omega_0\cos\omega_0 t + \omega_0 e^{-\gamma t}\right\}. \checkmark \tag{3.14}$$

4 Chapter 4 Solutions

1. Approximate Earth as a solid sphere of uniform density and radius $R = 6360$ km. Suppose you drill a tunnel from the north pole directly to another point on the surface described by a polar angle θ relative to the north pole. Drop a mass into the hole and let it slide through tunnel without friction. Find the frequency f with which the mass oscillates back and forth. Ignore Earth's rotation. Compare this to the frequency of a low-lying circular orbit.

   Solution:
   Choose the center of the tunnel as the center of the coordinate system, with $x = 0$. The force along the direction of the tunnel is

   $$F_x = mg(r)\frac{x}{r}.$$

   Because $g(r) = g(R)r/R$, where R is the Earth's radius,

   $$F_x = mg(R)\frac{x}{R},$$

   and the angular frequency and period are:

   $$\begin{aligned} \omega &= \sqrt{\frac{mg(R)}{mR}} = \sqrt{9.8\ \mathrm{m/s^2}/6.37 \times 10^6\ \mathrm{m}} \\ T = \frac{2\pi}{\omega} &= 5063\ \text{seconds} = 84\ \text{minutes}. \end{aligned}$$

   The frequency is then

   $$f = 1/T = 0.0119\ \text{oscillations/minute}.$$

   All tunnels can take you from any point to any other point on the planet in $T/2 = 42$ minutes.

2. Consider the gravitational field of the moon acting on the Earth.

   (a) Calculate the term k in the expansion

   $$g_{\text{moon}} = g_0 + kz + \cdots,$$

   where z is measured relative to Earth's center and is measured along the axis connecting the Earth and moon. Give your answer in terms of the distance between the moon and the earth, R_m and the mass of the moon M_m.

   (b) Calculate the difference between the height of the oceans between the maximum and minimum tides. Express your answer in terms of the quantities above, plus Earth's radius, R_e. Then give you answer in meters.

Solution:

$$k = \partial_R \frac{GMm}{R^2} = -\frac{2GMm}{R^3},$$

where R is the distance between the Earth and moon. The surface of the ocean should be at equipotential,

$$\frac{1}{2}kz^2 + mgh = \text{Constant} = \frac{1}{2}kR_{\text{earth}}^2 \cos^2\theta + mgh(\theta).$$

This gives

$$h(\theta = 0) - h(\theta = \pi/2) = \frac{1}{2}kR_{\text{earth}}^2/mg = \frac{GMR_{\text{earth}}^2}{gR^3} = 0.70 \text{ m}.$$

3. Consider an ellipse defined by the sum of the distances from the two foci being $2D$, which expressed in a Cartesian coordinates with the middle of the ellipse being at the origin becomes
$$\sqrt{(x-a)^2 + y^2} + \sqrt{(x+a)^2 + y^2} = 2D.$$
Here the two foci are at $(a, 0)$ and $(-a, 0)$. Show that this form is can be written as
$$\frac{x^2}{D^2} + \frac{y^2}{D^2 - a^2} = 1.$$

   Solution:

$$\begin{aligned}
4D^2 &= 2(x^2+y^2) + 2a^2 \\
&\quad +[(x-a)^2+y^2]^{1/2}[(x+a)^2+y^2]^{1/2} \\
\left[(2D^2-a^2)-(x^2+y^2)\right]^2 &= [(x-a)^2+y^2][(x+a)^2+y^2] \\
(x^2+y^2) - 2(2D^2-a^2)(x^2+y^2) + (2D^2-a^2)^2 &= (x^2+y^2+a^2)^2 - 4x^2a^2 \\
(x^2+y^2)\left(2(2D^2-a^2)+2a^2\right) - 4x^2a^2 &= 4D^4 - 4D^2a^2 \\
x^2(4D^2-4a^2) + y^2(4D^2) &= 4(D^2-a^2)D^2 \\
\frac{x^2}{D^2} + \frac{y^2}{D^2-a^2} &= 1.
\end{aligned}$$

4. Consider a particle in an attractive inverse-square potential, $U(r) = -\alpha/r$, where the point of closest approach is $r_{\min}$ and the total energy of the particle is E. Find the parameter A describing the trajectory in Eq. (**??**). Hint: Use the fact that at $r_{\min}$ there is no radial kinetic energy and $E = -\alpha/r_{\min} + L^2/2mr_{\min}^2$.

   Solution:

$$\begin{aligned} r &= \frac{1}{(m\alpha/L^2) + A\cos(\theta - \theta_0)}, \\ r_{\rm min} &= \frac{1}{(m\alpha/L^2) + A}, \\ E &= \frac{L^2}{2mr_{\rm min}^2} - \frac{\alpha}{r_{\rm min}}, \\ L^2 &= 2mEr_{\rm min}^2 + 2m\alpha r_{\rm min} \end{aligned}$$

The last expression was found by solving the quadratic equation for $1/r_{\rm min}$. Now, one can solve for A

$$\begin{aligned} A &= \frac{1}{r_{\rm min}} - \frac{m\alpha}{L^2} \\ &= \frac{1}{r_{\rm min}} - \frac{\alpha}{2Er_{\rm min}^2 + 2\alpha r_{\rm min}} \end{aligned}$$

This gives

$$\begin{aligned} r &= \frac{1}{(m\alpha)/(2mEr_{\rm min}^2 + 2m\alpha r_{\rm min}) + [1/r_{\rm min} - \alpha/(2Er_{\rm min}^2 + 2\alpha r_{\rm min})]\cos\theta} \\ &= \frac{r_{\rm min}}{\alpha/(2Er_{\rm min} + 2\alpha) + [1 - \alpha/(2Er_{\rm min} + 2\alpha)]\cos\theta} \\ &= r_{\rm min}\frac{2(E + \alpha/r_{\rm min})}{\alpha/r_{\rm min} + [2E + \alpha/r_{\rm min}]\cos\theta}. \end{aligned}$$

5. Consider the effective potential for an attractive inverse-square-law force, $F = -\alpha/r^2$. Consider a particle of mass m with angular momentum L.

   (a) Find the radius of a circular orbit by solving for the position of the minimum of the effective potential.

   (b) What is the angular frequency, $\dot{\theta}$, of the orbit? Solve this by setting $F = m\dot{\theta}^2 r$.

   (c) Find the effective spring constant for the particle at the minimum.

   (d) What is the angular frequency for small vibrations about the minimum? How does the compare with the answer to (b)?

Solution:
a)

$$\begin{aligned} \frac{d}{dr}\left(\frac{L^2}{2mr^2} - \frac{\alpha}{r}\right) &= 0, \\ \frac{L^2}{mr_{\rm min}^3} &= \frac{\alpha}{r_{\rm min}^2}, \\ r_{\rm min} &= \frac{L^2}{m\alpha}. \end{aligned}$$

b)

$$\begin{aligned} \frac{\alpha}{r_{\rm min}^2} &= m\dot\theta^2 r_{\rm min}, \\ \dot\theta &= \sqrt{\frac{\alpha}{m r_{\rm min}^3}} = \sqrt{\frac{m^2\alpha^4}{L^6}} = \frac{m\alpha^2}{L^3}. \end{aligned}$$

c)

$$k_{\rm eff} = \frac{d^2}{dr^2}U_{\rm eff} = 3\frac{L^2}{m r_{\rm min}^4} - 2\frac{\alpha}{r_{\rm min}^3} = \frac{\alpha}{r_{\rm min}^3} = \frac{m^3\alpha^4}{L^6}$$

d)

$$\omega = \sqrt{k_{\rm eff}/m} = \frac{m\alpha^2}{L^3}.$$

6. Consider a particle of mass m moving in a potential

$$U = \alpha \ln(r/a).$$

(a) If the particle is moving in a circular orbit of radius R, find the angular frequency $\dot\theta$. Solve this by setting $F = m\dot\theta^2 r$.

(b) Express the angular momentum L in terms of α, m and R. Also express R in terms of L, α and m.

(c) Sketch the effective radial potential, $V_{\rm eff}(r)$, for a particle with angular momentum L. (No longer necessarily moving in a circular orbit.)

(d) Find the position of the minimum of $V_{\rm eff}$ in terms of L, α and m, then compare to the result of (b).

(e) What is the effective spring constant for a particle at the minimum of $V_{\rm eff}$? Express your answer in terms of L, m and α.

(f) What is the angular frequency, ω, for small oscillations of r about the $R_{\rm min}$? Express your answer in terms of $\dot\theta$ from part (a).

Solution:

a)

$$\begin{aligned} F &= ma \\ -\frac{\alpha}{R} &= -m\dot\theta^2 R, \\ \dot\theta &= \sqrt{\frac{\alpha}{mR^2}}. \end{aligned}$$

b)

$$\begin{aligned} L &= mR^2\dot\theta = \sqrt{mR^2\alpha} \\ R &= \sqrt{\frac{L^2}{m\alpha}}. \end{aligned}$$

c)

$$V_{\text{eff}} = \frac{L^2}{2mr^2} + \alpha \ln r.$$

d)

$$\begin{aligned}
\frac{d}{dr} V_{\text{eff}} &= 0 \\
-\frac{L^2}{mr_{\min}^3} + \frac{\alpha}{r_{\min}} &= 0, \\
r_{\min} &= \sqrt{\frac{L^2}{m\alpha}}, \quad \checkmark.
\end{aligned}$$

e)

$$k_{\text{eff}} = -\frac{\alpha}{r_{\min}^2} + 3\frac{L^2}{mr_{\min}^4} = 2\frac{m\alpha^2}{L^2}$$

f)

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \frac{\alpha}{L}\sqrt{2} = \dot{\theta}\sqrt{2}.$$

7. Consider a particle of mass m in an attractive potential, $U(r) = -\alpha/r$, with angular momentum L with just the right energy so that

$$A = m\alpha/L^2$$

where A comes from the expression

$$r = \frac{1}{(m\alpha/L^2) + A\cos\theta}.$$

The trajectory can then be rewritten as

$$r = \frac{2r_0}{1 + \cos\theta}, \quad r_0 = \frac{L^2}{2m\alpha}.$$

(a) Show that for this case the total energy E approaches zero.

(b) Write this trajectory in a more recognizable parabolic form,

$$x = x_0 - \frac{y^2}{R}.$$

I.e., express x_0 and R in terms of r_0.

(c) Explain how a particle with zero energy can have its trajectory not go through the origin.

(d) What is the scattering angle for this trajectory?

Solution:
a) Substitute for r_0 in the first term for the energy below,

$$E = \frac{L^2}{2mr_0^2} - \frac{\alpha}{r_0} = \frac{\alpha}{r_0} - \frac{\alpha}{r_0}$$

b)

$$\begin{aligned} r(1+\cos\theta) &= 2r_0, \\ r+x &= 2r_0, \\ r^2 &= (2r_0-x)^2 = 4r_0^2 - 4r_0x + x^2 = x^2+y^2, \\ y^2 &= 4r_0^2 - 4r_0x, \\ x &= r_0 - \frac{y^2}{4r_0}. \end{aligned}$$

c) The energy $E \to 0$, but the combination $p_0^2b^2 = L^2$ is finite, i.e. $b \to \infty$. For a finite b the particle would indeed go through the origin. One way to put it, is that a particle on this trajectory never gets to infinity, so the questions is somewhat moot.

d) The angles for which $r \to \infty$ are $\theta' = \pm\pi$, so $\theta_s = \pi$.

8. Show that if one transforms to a reference frame where the total momentum is zero, $\vec{p}_1 = -\vec{p}_2$, that the relative momentum $\vec{q}$ corresponds to either $\vec{p}_1$ or $-\vec{p}_2$. This means that in this frame the magnitude of $\vec{q}$ is one half the magnitude of $\vec{p}_1 - \vec{p}_2$.

   Solution:
   Boost by the center-of-mass velocity.

$$\vec{q} = \mu(\vec{v_1} - \vec{v_2}) = \mu\left(\frac{\vec{p}_1}{m_1} - \frac{\vec{p}_2}{m_2}\right).$$

   If $\vec{p}_2 = -\vec{p}_1$,

$$\vec{q} = \mu\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{p}_1 = \vec{p}_1.$$

   One can repeat by substituting for $\vec{p}_1$ instead, then $\vec{q} = \vec{p}_2$.

9. Given the center of mass coordinates $\vec{R}$ and $\vec{r}$ for particles of mass m_1 and m_2, find the coordinates $\vec{r}_1$ and $\vec{r}_2$ in terms of $\vec{R}$ and $\vec{r}$ and the masses.

   Solution:

$$\begin{aligned} r &= r_1 - r_2, \\ R &= \frac{1}{m_1+m_2}(m_1r_1 + m_2r_2) \\ r + R\frac{m_1+m_2}{m_2} &= r_1\left(1 + \frac{m_1}{m_2}\right) = r_1\frac{m_1+m_2}{m_2}, \\ r_1 &= \frac{r + R(m_1+m_2)/m_2}{(m_1+m_2)/m_2} \\ &= R + \frac{m_2}{m_1+m_2}r. \end{aligned}$$

Similarly

$$r_2 = R - \frac{m_1}{m_1 + m_2} r.$$

10. Consider two particles of identical mass scattering at an angle $\theta_{\rm cm}$ in the center of mass.

    (a) In a frame where one is the target (initially at rest) and one is the projectile, find the scattering angle in the lab frame, θ, in terms of $\theta_{\rm cm}$.

    (b) Express $d\sigma/d\cos\theta$ in terms of $d\sigma/d\cos\theta_{\rm cm}$. I.e., find the Jacobian for $d\cos\theta_{\rm cm}/d\cos\theta$ in terms of $\cos\theta_{\rm cm}$.

Solution:
In the center-of-mass frame (primed momenta)

$$\begin{aligned} p'_x &= p_x - mV_{\rm cm} = p_x - |p_0|/2, \\ p'_y &= p_y. \end{aligned}$$

Here p_0 is the initial momentum in the lab frame which equals $2p'$. Solving for p_x,

$$\begin{aligned} p_x &= p'(1+\cos\theta') \\ \cos\theta &= \frac{p_x}{\sqrt{p_x^2 + p_y^2}} \\ &= \frac{p'(1+\cos\theta')}{\sqrt{p'^2(1+\cos\theta')^2 + p'^2\sin^2\theta'}} \\ &= \sqrt{(1+\cos\theta')/2}, \\ \frac{d\cos\theta}{d\cos\theta'} &= \frac{1}{2\sqrt{2(1+\cos\theta')}}, \\ \frac{d\cos\theta'}{d\cos\theta} &= 2\sqrt{2(1+\cos\theta')}. \end{aligned}$$

11. Assume you are scattering alpha particles (He-4 nuclei $Z = 2, A = 4$) off of a gold target ($Z = 79, A = 197$). If the radius of the nucleus is 7.5×10^{-15} meters, and if the energy of the beam is 38 keV,

    (a) What is the total cross section for having a nuclear collision? Give the answer in millibarns, 1 mb$= 10^{-31}$ m^2.

    (b) Find the scattering angle (in degrees) at which the Rutherford differential cross section formula breaks down?

Solution:
a) Use conservation of L to find the angular momentum at R,

$$\begin{aligned} \frac{L^2}{2mR^2} - \frac{\alpha}{R} &= E, \\ L^2 &= 2mR^2\left(E + \frac{\alpha}{R}\right). \end{aligned}$$

The impact parameter can be found using $L = p_0 b$,

$$b^2 = \frac{L^2}{p_0^2} = \frac{L^2}{2mE} = R^2 \frac{E + \frac{\alpha}{R}}{E},$$
$$\sigma = \pi b^2 = \pi R^2 \frac{E + \frac{\alpha}{R}}{E}.$$

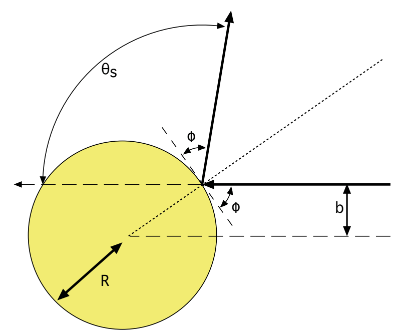
Plugging in numbers $\sigma = 1400$ barns

b) Use Eq. (4.37)

$$\sin(\theta_s/2) = \frac{a}{\sqrt{a^2 + b^2}}, \quad a \equiv \frac{\alpha}{2E}, \quad b = \text{from step a.} \tag{4.1}$$

Plugging in numbers, $\theta_s = 83$ degrees.

12. A point particle is fired at a spherical target of radius R. The particle bounces off the target elastically with scattering angle θ_s. The angle ϕ in the figure is only meant to show that for a plane tangent to the surface, the angles relative to the surface are equal for the incoming and outgoing trajectories.



(a) Find the differential cross section $d\sigma/d\Omega = (1/2\pi)(d\sigma/d\cos\theta_s)$.

(b) Integrate $d\sigma/d\Omega$ to obtain the total cross section.

Solution:

a) First relate b to θ_s. From trignometry

$$b = R\sin\left[\frac{(\pi - \theta_s)}{2}\right],$$
$$|d\sigma| = 2\pi b db = 2\pi R^2 \sin\left[\frac{(\pi - \theta_s)}{2}\right] \cos\left[\frac{(\pi - \theta_s)}{2}\right] \frac{d\theta_s}{2}$$
$$=]\frac{1}{2}\pi R^2 \sin\left[(\pi - \theta_s)\right] d\theta_s$$
$$= \tfrac{\pi}{2} R^2 d\cos\theta_s,$$
$$\frac{d\sigma}{2\pi d\cos\theta_s} = \tfrac{R^2}{4}.$$

b) Integrating the constant,

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \pi R^2 \quad \checkmark \tag{4.2}$$

5 Chapter 5 Solutions

1. Consider a pail of water spinning about a vertical axis at the center of the pail with frequency ω. Find the height of the water (within a constant) as a function the radius $r_\perp$ from the axis of rotation. Use the concept of a centrifugal potential in the rotating frame.

   Solution:
   The effective potential for a drop of mass δm depends on its position,

   $$\delta PE = -\frac{1}{2}\delta m\omega^2 r^2 + \delta mgy.$$

   This must be constant on the surface, $y \to h(r)$,

   $$\begin{aligned} \delta mgh(r) &= \frac{1}{2}\delta m\omega^2 r^2, \\ h(r) &= \frac{1}{2}\frac{\omega^2 r^2}{g}. \end{aligned}$$

2. A high-speed cannon shoots a projectile with an initial velocity of 1000 m/s in the east direction. The cannon is situated in Minneapolis (latitude of 45 degrees) The projectile velocity is nearly horizontal and it hits the ground after a distance $x = 3000$ m. Find the alteration of the point of impact in the north-south (y) direction due to the Coriolis force. Assume the effect is small so that you can approximate the eastward (x) component of the velocity as being constant. Be sure to indicate whether the deflection is north or south.

   Solution:
   Let y be the north-south direction,

   $$\begin{aligned} \frac{dv_y}{dt} &= -2\omega_z v_x + 2\omega_x v_z, \\ \omega_z = \omega_{\rm earth}/\sqrt{2},\;\; \omega_x = 0, & \\ v_y &\approx -2\omega_{\rm earth} v_x t/\sqrt{2}, \\ \delta y &\approx -\omega_{\rm earth} v_x t^2/\sqrt{2} \\ &\approx -\frac{\omega_{\rm earth} x^2}{v_x\sqrt{2}}. \end{aligned}$$

   This comes out to 0.46 meters south. The approximations are because you assume v_x is constant, whereas it does change ever so slightly due to the Coriolis force.

3. Someone wishes to use a Foucault pendulum as a crude clock. If the person lives in Minneapolis, how much time will pass between having the pendulum swinging in the east-west direction until it swings in the north-south direction.

   Solution:
   From the Foucalt example in the lecture notes

   $$\begin{aligned} \Omega_z &= |\Omega|/\sqrt{2}, \\ T &= \frac{2\pi}{\Omega_z} = \frac{2\pi}{(2\pi/(24 \text{ hours}))}\sqrt{2} \end{aligned}$$

The time required is one fourth a period or 6 hours $\times\sqrt{\mathbf{2}} \approx \mathbf{8.5}$ hours.

6 Chapter 6 Solutions

1. Consider a hill whose height y is given as a function of the horizontal coordinate x. Consider a segment of the hill from $x = 0$ to $x = L$ with initial height $y(x = 0) = 0$ and whose final height is $y(x = L) = -h$. Transforming the last equation in Example for a downward vertical force rather than a horizontal force,

$$x = -\sqrt{-2ay - y^2} + a \arccos(1 + y/a).$$

Consider a wheel of radius a rolling along the bottom of the x axis. Mark a point on the top of the wheel, which is originally at the origin, $x = y = 0$, when the top of the wheel touches the origin. As the wheel rolls by an angle θ the marked point moves due to both the translation and the rotation of the wheel. The y coordinate of the marked point is

$$y = -a(1 - \cos\theta),$$

whereas the x coordinate is

$$x = a\theta - a\sin\theta.$$

The first term is due to the horizontal translation of the axis, while the second term arises from the rotation of the wheel. Re-express these two equations to find $x(y)$.

$$\begin{aligned}
\cos\theta &= 1 + \frac{y}{a}, \\
\theta &= \arccos\left(1 + y/a\right), \\
x &= a\arccos\left(1 + y/a\right) - a\sqrt{1 - \cos^2\theta} \\
&= a\arccos\left(1 + y/a\right) - a\sqrt{-2\frac{y}{a} - \frac{y^2}{a^2}} \\
&= -\sqrt{-2ay - y^2} + a\arccos\left(1 + y/a\right).
\end{aligned}$$

2. Consider a chain of length L that hangs from two supports of equal height stretched from $x = -X$ to $x = +X$. The general solution for a catenary is

$$y = \lambda + a\cosh[(x - x_0)/a],$$

   (a) Using symmetry arguments, what is x_0.

   (b) Express the length L in terms of X and a.

   (c) Numerically solve the transcendental equation above to find a in terms of $L = 10$ m and $X = 4$ m.

Solution:
a) $x_0 = 0$
b)

$$L = 2\int_0^X dx\,\sqrt{1 + \sinh^2(x/a)} = 2a\sinh(X/a).$$

c) Program based on what was done in class. Note as $a \to \infty$ that $L = 2X$ as expected.

3. Consider a mass m connected to a spring with spring constant k. Rather than being fixed, the other end of the spring oscillates with frequency ω and amplitude A. For a generalized coordinate, use the displacement of the mass from its relaxed position and call it $y = x - \ell - A\cos\omega t$.

   (a) Write the kinetic energy in terms of the generalized coordinate.

   (b) Write down the Lagrangian.

   (c) Find the equations of motion for y.

   Solution:

$$\begin{aligned} a)\ T &= \frac{1}{2}m\left(\dot{y} + \omega A \sin\omega t\right)^2 \\ b)\ L &= \frac{1}{2}m\left(\dot{y} + \omega A \sin\omega t\right)^2 - \frac{1}{2}ky^2 \\ c)\ m\frac{d}{dt}\left(\dot{y} + \omega A \sin\omega t\right) &= -ky, \\ m\ddot{y} + m\omega^2 A\cos\omega t &= -ky. \end{aligned}$$

   This looks like a driven harmonic oscillator

$$m\ddot{y} + ky = m\omega^2 A\cos\omega t.$$

4. Consider a bead of mass m on a circular wire of radius R. Assume a force kx acts on the spring, where x is measured from the center of the circle. Using θ as the generalized coordinate (measured relative to the x axis),

   (a) Write the Lagrangian in terms of θ.

   (b) Find the equations of motion.

   Solution:

$$\begin{aligned} a)\ T &= \frac{1}{2}mR^2\dot{\theta}^2, \\ V &= -\frac{1}{2}k(x-R)^2 = 1\frac{1}{2}R^2\cos^2\theta, \\ L &= \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}kR^2\cos^2\theta. \\ b)\ mR^2\ddot{\theta} &= -kR^2\sin\theta\cos\theta, \\ \ddot{\theta} &= -\omega_0^2\sin\theta\cos\theta, \quad \omega_0^2 \equiv k/m. \end{aligned}$$

5. Consider a pendulum of length ℓ with all the mass m at its end. The pendulum is allowed to swing freely in both directions. Using ϕ to describe the azimuthal angle about the z axis and θ to measure the angular deviation of the pendulum from the downward direction, address the following questions:

(a) If the pendulum is initially moving horizontally with velocity v_0 and angle $\theta_0 = 90°$ (horizontal), use energy and angular momentum conservation to find the minimum angles of $\theta_{\rm min}$ subtended by the pendulum. (Note that the angle will oscillate between $90°$ and the minimum angle.

(b) Write the Lagrangian using θ and ϕ as generalized coordinates.

(c) Write the equations of motion for θ and ϕ.

(d) Rewrite the equations of motion for θ using angular momentum conservation to eliminate and reference to ϕ.

(e) Find the value of L required for the stable orbit to be at $\theta = 45°$.

(f) For the steady orbit found in (e) consider small perturbations of the orbit. Find the frequency with which the pendulum oscillates around $\theta = 45°$.

Solution:

a) The kinetic energy from the θ motion disappears at the maximum and minimum angles. The remaining potential energy and kinetic energy due to $\dot{\phi}$ must be equal at these two angles.

$$\begin{aligned}\frac{1}{2}mv_0^2 &= -mg\ell\cos\theta_{\rm min} + \frac{1}{2}mv^2(\theta_{\rm min})\\ &= -mg\ell\cos\theta_{\rm min} + \frac{L^2}{2m\ell^2\sin^2\theta_{\rm min}}, \quad L = mv_0\ell\end{aligned}$$

Using the fact that

$$\frac{L^2}{2m\ell^2} = \frac{1}{2}mv_0^2,$$

the equation for $\theta_{\rm min}$ is then

$$\begin{aligned}-\frac{1}{2}mv_0^2\cos^2\theta_{\rm min} &= -mg\ell\cos\theta_{\rm min}\sin^2\theta_{\rm min},\\ \frac{1}{2}v_0^2\cos\theta_{\rm min} &= g\ell(1-\cos^2\theta_{\rm min}).\end{aligned}$$

One can now solve for $\theta_{\rm min}$,

$$\begin{aligned}\cos\theta_{\rm min} &= \sqrt{\alpha^2+1}-\alpha,\\ \alpha &= \frac{v_0^2}{4g\ell}.\end{aligned}$$

b)

$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 + \frac{1}{2}m\ell^2\sin^2\theta\dot{\phi}^2 + mg\ell\cos\theta.$$

c)

$$\begin{aligned}m\ell^2\ddot{\theta} &= -mg\ell\sin\theta + m\ell^2\sin\theta\cos\theta\dot{\phi}^2,\\ \ddot{\theta} &= -\frac{g}{\ell}\sin\theta + \dot{\phi}^2\sin\theta\cos\theta,\\ \frac{d}{dt}\left(m\ell^2\sin^2\theta\dot{\phi}\right) &= 0,\\ \sin^2\theta\dot{\phi} &= \text{constant}.\end{aligned}$$

d)

$$\begin{aligned}
\ddot{\theta} &= -\frac{g}{\ell}\sin\theta + \frac{L^2}{m^2\ell^4\sin^4\theta}\sin\theta\cos\theta, \\
\ddot{\theta} &= -\frac{g}{\ell}\sin\theta + \frac{L^2}{m^2\ell^4}\frac{\cos\theta}{\sin^3\theta}.
\end{aligned}$$

e)

$$\begin{aligned}
\frac{g}{\ell}\sin\theta &= \frac{L^2}{m^2\ell^4}\frac{\cos\theta}{\sin^3\theta}, \\
L^2 &= \frac{\sin^4\theta}{\cos\theta}(m^2 g\ell^3), \\
L &= \frac{1}{2^{3/2}}\sqrt{m^2 g\ell^2} \text{ for } \theta = 45°.
\end{aligned}$$

f)

$$\begin{aligned}
\ddot{\theta} &= -\frac{g}{\ell}\sin\theta + \frac{L^2}{m^2\ell^4}\frac{\cos\theta}{\sin^3\theta}, \\
\ddot{\theta} &= -\frac{g}{\ell}\cos\theta\delta\theta - \frac{L^2}{m^2\ell^4}\left(\frac{1}{\sin^2\theta} + 3\frac{\cos^2\theta}{\sin^4\theta}\right)\delta\theta, \\
\omega^2 &= \frac{g}{\ell}\frac{1}{\sqrt{2}} + 8\frac{L^2}{m^2\ell^4} \\
&= \frac{1}{\sqrt{2}}\frac{g}{\ell} + \frac{8L^2}{m^2\ell^4}.
\end{aligned}$$

6. Consider a mass m that is connected to a wall by a spring with spring constant k. A second identical mass m is connected to the first mass by an identical spring. Motion is confined to the x direction.

   (a) Write the Lagrangian in terms of the positions of the two masses x_1 and x_2.

   (b) Find the equations of motion.

   (c) Find two solutions of the type

   $$x_1 = Ae^{i\omega t}, \quad x_2 = Be^{i\omega t}.$$

   Solve for A/B and ω. Express your answers in terms of $\omega_0^2 = k/m$.

Solution:
a)

$$\begin{aligned}
T &= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2, \\
V &= \frac{1}{2}kx_1^2 + \frac{1}{2}(x_1 - x_2)^2, \\
L &= T - V.
\end{aligned}$$

b)

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 - k(x_1 - x_2) \\ &= -2kx_1 + kx_2, \\ m\ddot{x}_2 &= -k(x_2 - x_1). \end{aligned}$$

c)

$$\begin{aligned} x_1 &= Ae^{i\omega t}, \quad x_2 = Be^{i\omega t}, \\ -m\omega^2 A &= -2kA + kB, \\ -m\omega^2 B &= -kB + kA. \end{aligned}$$

Divide both sides by B

$$\begin{aligned} \omega^2 \left(\frac{A}{B}\right) &= 2\omega_0^2 \left(\frac{A}{B}\right) - \omega_0^2, \\ -\omega^2 &= -\omega_0^2 + \omega_0^2 \left(\frac{A}{B}\right), \\ \omega_0^2 &\equiv \frac{k}{m}. \end{aligned}$$

Treat A/B and ω^2 and ω^2 as unknowns. Solve the 2 eq.s. 2 unknowns and get

$$\begin{aligned} \omega^2 \left(1 - \frac{\omega^2}{\omega_0^2}\right) &= \omega_0^2 - 2\omega^2, \\ \omega^4 - 3\omega_0^2\omega^2 + \omega_0^4 &= 0, \\ \omega^2 &= \omega_0^2 \frac{3 \pm \sqrt{5}}{2}. \end{aligned}$$

The two values A/B are

$$A/B = \left(1 - \frac{\omega_\pm^2}{\omega_0^2}\right) = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}.$$

7. Consider two masses m_1 and m_2 interacting according to a potential $V(\vec{r}_1 - \vec{r}_2)$.
   (a) Write the Lagrangian in terms of the generalized coordinates $\vec{R}_{\rm cm} = (m_1\vec{r}_1 + m_2\vec{r}_2)/(m_1 + m_2)$ and $\vec{r} = \vec{r}_1 - \vec{r}_2$ and their derivatives.
   (b) Using the independence of L with respect to $\vec{R}_{\rm cm}$, identify a conserved quantity.

Solution:
a)

$$\begin{aligned} L &= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2, -V(\vec{r}), \\ M &= m_1 + m_2, \\ \mu &= \frac{m_1 m_2}{m_1 + m_2}. \end{aligned}$$

b)

$$\begin{aligned} \frac{d}{dt}\left(M\dot{\vec{R}}\right) &= 0, \\ \dot{\vec{R}} &= \text{constant}. \end{aligned}$$

The center of mass velocity is constant.