your names____

____

____

Exercise, Monday, Nov. 14, 2022

You wish to find the fastest route for a paddle boat, which moves with speed V_0 on still water, to travel a distance L, from $x = y = 0$ to $x = L, y = 0$. There is a current also moving in the x direction, but depending on y. The current's speed is $w_x = \alpha y$. Rather than minimizing the time to go a distance L, we will maximize the distance L one travels in a net time T. This will result in the same trajectory once the time is adjusted so the distance is indeed L. That distance is

$$L = \int_0^t dt' \frac{dx}{dt}(y, y'), \quad y' \equiv dy/dt$$

Our goal is use the Euler-Lagrange equation to solve for $y(t)$.

1. (10 pts) Express dx/dt in terms of w_x, V_0 and y'.

2. (10 pts) Using the expression for x above, write a 2nd-order differential equation for $y(t)$ from the Euler-Lagrange equations. The equation should be in terms of L, α and w_x, y, d^2y/dt^2 and dy/dt. Arrange answer in the form:

$$\frac{d^2y}{dt^2} = \text{function of everything else}$$

3. (20 extra credit quiz points – will probably take a few hours) Write a program to solve the Euler-Lagrange equation. Use $\alpha = 0.001\ \mathrm{s}^{-1}$, $T = 1$ hour, and $V_0 = 10$ m/s. What is the maximum value of y for this trajectory? What is the distance traveled in the x direction for this trajectory?
HINT: Might be less difficult to solve for $y' = dy/dt$ from mid-point of trajectory to end, using fact that (by symmetry) $y'(x = L/2) = 0$.

Extra-credit portion can be turned in Monday, which should include printout of program and the value of $y_{\max}$.

Solution:
a)

$$v_x(y, y'; t) = \alpha y + \sqrt{V_0^2 - y'^2}$$

b)

$$\begin{aligned}
\frac{d}{dt}\frac{\partial v_x}{\partial y'} &= \frac{\partial v_x}{\partial y} \\
\frac{d}{dt}\frac{-\alpha y'}{\sqrt{V_0^2 - y'^2}} &= \alpha \\
y''\left\{\frac{-\alpha}{\sqrt{V_0^2 - y'^2}} - \frac{\alpha y'^2}{(V_0^2 - y'^2)^{3/2}}\right\} &= \alpha \\
y'' &= -\alpha\frac{(V_0^2 - y'^2)^{3/2}}{V_0^2}.
\end{aligned}$$

Extra Credit:
To solve this first write equation as

$$\frac{dy'}{(V_0^2 - y'^2)^{3/2}} = -\alpha V_0^2 dt,$$

Integrate both sides. But to make life easier, work in terms of variable $t' = t - T/2$, because by symmetry $y'(t'=0) = y'(t=T/2) = 0$. Then

$$\int_0^{y'/V_0} \frac{du}{(1-u^2)^{3/2}} = -\alpha t'.$$

To solve numerically, this give t' for a given y'. One can then integrate y' to get $y(t')$. Because $y(t' = \pm T/2) = 0$, one subtracts $y'(t' = T/2)$ from the result for $y'(t')$ to get $y(t')$. Once one knows y, along with y' then one can express $v_x(t')$. Finally, one can integrate $v_x(t')$ to get $x'(t')$, then subtract $x'(-T/2)$ to get $x(t')$. This is not an easy program to write.

Alternatively, one can also solve for all the relations analytically.

$$\begin{aligned}\frac{u}{\sqrt{1-u^2}} &= -\alpha t', \quad u = y'/V_0 \\ y'(t') &= -V_0 \frac{\alpha t}{\sqrt{1+\alpha^2 t'^2}}, \\ t' &= t - T/2.\end{aligned}$$

Next, integrate $y'(t')$ to get $y(t')$. Choose arbitrary constant to ensure that $y(t' = -T/2) = y(t' = T/2) = 0$.

$$\begin{aligned}y(t') &= \frac{-V_0}{\alpha}\sqrt{1+\alpha^2 t'^2} + \frac{V_0}{\alpha}\sqrt{1+\alpha^2 (T/2)^2}, \\ t' &= t - T/2.\end{aligned}$$

Next, write down v_x,

$$\begin{aligned}v_x(t') &= \alpha y + V_0\sqrt{1 - \frac{\alpha^2 t'^2}{1+\alpha^2 t'^2}}, \\ &= V_0\sqrt{1+\alpha^2(T/2)^2} + V_0\left\{-\sqrt{1+\alpha^2 t'^2} + \sqrt{1-\alpha^2 t'^2/(1+\alpha^2 t'^2)}\right\} \\ &= V_0\sqrt{1+\alpha^2(T/2)^2} + V_0\left\{-\sqrt{1+\alpha^2 t'^2} + 1/\sqrt{1+\alpha^2 t'^2}\right\}, \\ t' &= t - T/2.\end{aligned}$$

Finally, integrate v_x to get $x(t')$. But first do an integral

$$\begin{aligned}\int du\ \sqrt{1+u^2} &= \int d\theta \cosh^2\theta, \quad u = \sinh\theta, \\ &= \frac{1}{2}\int d\theta(1+\cosh 2\theta) = \frac{1}{2}\left[\theta + \sinh 2\theta/2\right] \\ \int du\ \sqrt{1+u^2} &= \frac{1}{2\alpha}\left[\sinh^{-1}(\alpha t') + \alpha t'\sqrt{1+\alpha^2 t'^2}\right].\end{aligned}$$

The other integral yields an inverse sinh, and

$$x'(t') = V_0 t' \sqrt{1 + \alpha^2 (T/2)^2} + \frac{V_0}{2\alpha} \sinh^{-1}(\alpha t') - \frac{V_0 t'}{2} \sqrt{1 + \alpha^2 t'^2}$$

The quantity x' is the distance from the position at $T/2$. To get x, one must add back in $x'(t' = T/2)$. Here, $x(t = T) = 2x'(t' = T/2)$,

$$\begin{aligned} x &= x'(t') + x(t = T)/2, \\ t' &= t - T/2, \\ x(t = T) &= \frac{V_0 T}{2} \sqrt{1 + \alpha^2 (T/2)^2} + \frac{V_0}{\alpha} \sinh^{-1}(\alpha T/2). \end{aligned}$$