your name____________________________________

Physics 321 Practice Exam #2 - Wednesday, Nov. 19

FYI: For the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

the solutions are

$$\begin{aligned}
x &= A_1 e^{-\beta t}\cos\omega' t + A_2 e^{-\beta t}\sin\omega' t \quad \omega' = \sqrt{\omega_0^2 - \beta^2} \quad \text{(under damped)}\\
x &= A e^{-\beta t} + B t e^{-\beta t}, \quad \text{(critically damped)}\\
x &= A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}, \quad \beta_i = \beta \pm \sqrt{\beta^2 - \omega_0^2}, \quad \text{(over damped)}.
\end{aligned}$$

Coriolis and centrifugal forces

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}_{\rm real} - m\vec{\omega}\times\vec{\omega}\times\vec{r} - 2m\vec{\omega}\times\vec{v}.$$

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1. A small particle of mass m is aimed at a heavy target. They are attracted by a potential,

$$V(r) = -\frac{\alpha}{r^4}.$$

Find the cross sectional area for impacting the origin if the incoming energy of the particle is E.

Solution:
First find maximum height of the effective potential with angular momentum L,

$$\begin{aligned}
V_{\rm eff} &= -\frac{\alpha}{r^4} + \frac{L^2}{2mr^2}, && (1)\\
\frac{d}{dr}V_{\rm eff} &= 0 = \frac{4\alpha}{r^5} - \frac{L^2}{mr^3}, && (2)\\
r^2 &= \frac{4m\alpha}{L^2}, && (3)\\
V_{\rm max} &= -\frac{\alpha L^4}{16m^2\alpha^2} + \frac{L^4}{2m4m\alpha} && (4)\\
&= \frac{L^4}{16m^2\alpha}. && (5)
\end{aligned}$$

Any energy below $V_{\rm max}$ will be swallowed,

$$\begin{aligned}
E &> \frac{L^4}{16m^2\alpha} && (6)\\
E &> \frac{(2mEb^2)^2}{16m^2\alpha} && (7)\\
b^2 &< 2\sqrt{\frac{\alpha}{E}}, && (8)\\
\sigma &= \pi b^2 = 2\pi\sqrt{\frac{\alpha}{E}}. && (9)
\end{aligned}$$

2. A particle is in a circular orbit with angular velocity $\dot{\theta}$ due to a potential

$$V(r) = V_0 \ln(r/a).$$

If the radius is given a small perturbation, what is the frequency ω with which the particle's radius oscillates about the original value?

Solution:

$$\begin{aligned}
V_{\text{eff}} &= \frac{L^2}{2mr^2} + V_0 \ln(r/a), \\
\frac{d}{dr} V_{\text{eff}} &= -\frac{L^2}{mr^3} + \frac{V_0}{r} = 0, \quad \text{at min} \\
r_{\text{min}}^2 &= \frac{L^2}{mV_0} = \frac{m^2 r_{\text{min}}^4 \dot{\theta}^2}{mV_0}, \\
r_{\text{min}}^2 &= \frac{V_0}{m\dot{\theta}^2}.
\end{aligned}$$

The effective spring constant is the curvature at the minimum

$$\begin{aligned}
k_{\text{eff}} &= \frac{d^2}{dr^2} V_{\text{eff}} \\
&= \frac{3L^2}{mr_{\text{min}}^4} - \frac{V_0}{r_{\text{min}}^2} \\
&= 2m\dot{\theta}^2 \\
\omega &= \sqrt{k/m} = \dot{\theta}\sqrt{2}.
\end{aligned}$$

3. A particle is fired directly upward from a point on the equator with muzzle velocity $v_0 = 500$ m/s. Neglecting air resistance, where does the particle land relative to the firing point. Take into account the Coriolis force.

Solution:
Since the force is proportional to the velocity, and since for each time step δt the there will be equal and opposite force on the upward and downward tracks. The component v_x will then be a maximum at the top of the trajectory and return to zero at the bottom. Thus, for each element of the upward trajectory, the acceleration will be opposite on the way downward and v_x will be the SAME as it was on the way up. The final δx will then be twice the value at the top. To calculate δx on the way to the top,

$$\begin{aligned} a &= -2v_z\omega_0, & (10) \\ x \text{ points east} &= -2(v_0 - gt)\omega_0 & (11) \\ v_x &= -2\omega_0(v_0 t - gt^2/2), & (12) \\ \delta x &= -\ 2\omega_0(v_0 t^2/2 - gt^3/6), & (13) \\ \delta x &= -(2/3)\omega_0\frac{v_0^3}{g^2}, \text{ using } t = v_0/g & (14) \\ 2\delta x &= -(4/3)\omega_0\frac{v_0^3}{g^2} = 126 \text{ m.} & (15) \end{aligned}$$

It falls to the west from where it was launched.

4. Additionally, one problem on the midterm will be a reprise from a previous quiz or midterm.