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Physics 321 FINAL - Tuesday, Dec. 9

FYI: For the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

the solutions are

$$\begin{aligned} x &= A_1 e^{-\beta t}\cos\omega' t + A_2 e^{-\beta t}\sin\omega' t \quad \omega' = \sqrt{\omega_0^2 - \beta^2} \quad \text{(under damped)} \\ x &= A e^{-\beta t} + B t e^{-\beta t}, \quad \text{(critically damped)} \\ x &= A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}, \quad \beta_i = \beta \pm \sqrt{\beta^2 - \omega_0^2}, \quad \text{(over damped).} \end{aligned}$$

Coriolis and centrifugal forces

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}_{\text{real}} - m\vec{\omega} \times \vec{\omega} \times \vec{r} - 2\vec{\omega} \times \vec{v}.$$

Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

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1. (5 pts) A mass m hangs from a spring of spring constant k and comes to equilibrium. A small external force, $F_y = F_0 \sin \omega t$ is applied to the spring beginning at time $t = 0$. Find the vertical displacement $y(t)$.

Solution:
Find the particular solution

$$\begin{aligned} y_p &= Ce^{i\omega t}, \\ \frac{d^2 y_p}{dt^2} + \omega_0^2 y_p &= -i\frac{F_0}{m} e^{i\omega t}, \\ -\omega^2 C + \omega_0^2 C &= -i\frac{F_0}{m}, \\ C &= -i\frac{F_0/m}{\omega_0^2 - \omega^2}, \end{aligned}$$

$$\Re y_p = \frac{F_0/m}{\omega_0^2 - \omega^2} \sin \omega t.$$

The general solution, then applying the B.C.

$$\begin{aligned} y(t) &= A\cos\omega_0 t + B\sin\omega_0 t + \frac{F_0/m}{\omega_0^2 - \omega^2}\sin\omega t, \\ y(t=0) &= 0 = A, \\ v_y(t=0) &= 0 = B\omega_0 + \frac{\omega F_0/m}{\omega_0^2 - \omega^2} \\ y(t) &= \left(\sin\omega t - \frac{\omega}{\omega_0}\sin\omega_0 t\right)\frac{F_0/m}{\omega_0^2 - \omega^2} \end{aligned}$$

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2. (5 pts). A solid sphere of mass m and radius R is released at time $t = 0$ and falls according to the forces of gravity and a drag force $-\gamma A v^2$. Solve for the velocity as a function of time.

Solution:
Defining downward as positive

$$\begin{aligned}
\frac{dv}{dt} &= g - \eta v^2, \quad \eta \equiv \gamma A/m, \\
t &= \int_0^v \frac{dv'}{g - (\gamma A/m)v'^2} \\
&= \frac{1}{(\gamma A g/m)^{1/2}} \int_0^{v/v_{\rm max}} \frac{du}{1-u^2}, \quad v_{\rm max} = \sqrt{mg/\gamma A}, \\
t &= \frac{1}{(\gamma A g/m)^{1/2}} \tanh^{-1}(v/v_{\rm max}), \\
v &= v_{\rm max} \tanh\left(\gamma A g/m)^{1/2} t\right), \\
&= v_{\rm max} \tanh\left(\frac{gt}{v_{\rm max}}\right).
\end{aligned}$$

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3. (5 pts) A particle feels an attractive central potential,

$$V(r) = \beta r^{1/4}.$$

The particle is in a stable circular orbit of angular frequency ω_0, when it feels a small perturbation which causes the the radius r to oscillate about the original orbit's radius with frequency ω. Find ω in terms of ω_0.

Solution:

$$\begin{aligned}
V_{\rm eff} &= \beta r^{1/4} + \frac{L^2}{2mr^2}, && (1)\\
\left.\frac{\partial V_{\rm eff}}{\partial r}\right|_{r_0} &= 0 = \frac{1}{4}\beta r_0^{-3/4} - \frac{L^2}{mr_0^3}, && (2)\\
r_0^{9/4} &= \frac{4L^2}{\beta m}, \quad \omega_0 = \frac{L}{mr_0^2} && (3)\\
k_{\rm eff} &= \left.\frac{\partial^2 V_{\rm eff}}{\partial r^2}\right|_{r_0} = -\frac{3}{16}\beta r_0^{-7/4} + \frac{3L^2}{mr_0^4}, && (4)\\
&= -\frac{3}{16}\beta\frac{4L^2}{\beta m r_0^4} + \frac{3L^2}{mr_0^4} && (5)\\
&= \frac{9}{4}\frac{L^2}{mr_0^4}, && (6)\\
\omega^2 &= \frac{k_{\rm eff}}{m} = \frac{9}{4}\omega_0^2, && (7)\\
\omega &= \frac{3}{2}\omega_0. && (8)
\end{aligned}$$

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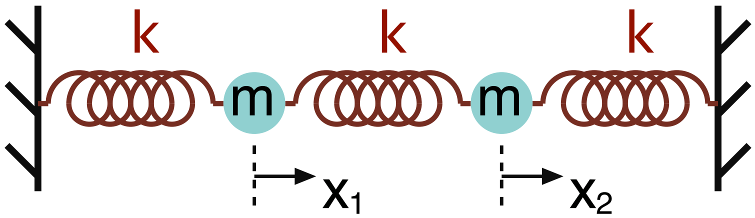


4. Consider the two identical masses connected to the three identical springs pictured above. Let x_1 and x_2 describe the displacement of the two masses relative to the equilibrium position.

   (a) (4 pts) Write the Lagrangian in terms of x_1 and x_2, then find the equations of motion.

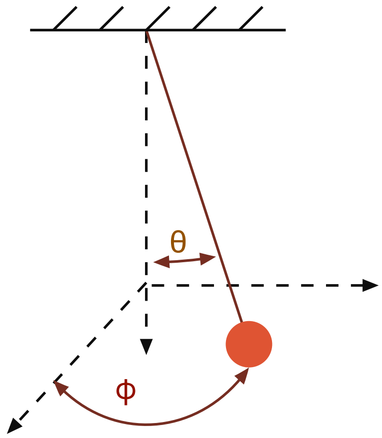
   (b) (4 pts) Assume there are solutions of the form,

   $$x_1 = Ae^{i\omega t}, \quad x_2 = Be^{i\omega t}.$$

   Find the frequencies of the two normal modes.

Solution:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}k(x_1 - x_2)^2 - \frac{1}{2}kx_2^2, \\
m\ddot{x}_1 &= -2kx_1 + kx_2, \\
m\ddot{x}_2 &= -2kx_2 + kx_1, \\
x_1 &= Ae^{i\omega t}, \quad x_2 = Be^{i\omega t}, \\
-\omega^2 A &= -2\omega_0^2 A + \omega_0^2 B, \\
-\omega^2 B &= -2\omega_0^2 B + \omega_0^2 A \\
A &= B\frac{2\omega_0^2 - \omega^2}{\omega_0^2}, \\
(2\omega_0^2 - \omega^2)\left(\frac{2\omega_0^2 - \omega^2}{\omega_0^2}\right) B &= \omega_0^2 B, \\
\omega^4 - 4\omega_0^2\omega^2 + 3\omega_0^4 &= 0, \\
\omega^2 &= \omega_0^2 \frac{4 \pm \sqrt{16 - 12}}{2}, \\
&= \omega_0^2(2 \pm 1), \\
\omega &= \omega_0, \text{ or } \omega_0\sqrt{3}.
\end{aligned}$$



5. Consider the swinging pendulum of mass m and length ℓ pictured above.

   (a) (4 pts) If the pendulum initially has a horizontal velocity v_0 and is at angle $\theta_0 = 90°$, find the minimum polar angle, $\theta_{\rm min}$, subtended by the trajectory of the pendulum.

   (b) (4 pts) Write the Lagrangian in terms of θ and ϕ, then find the equations of motion for θ using the angular momentum L to eliminate any mention of ϕ or $\dot\phi$.

Solution:

a) Use energy conservation plus the fact that $\dot\theta = 0$ initially and at lowest angle

$$\begin{aligned}
E &= \frac{L^2}{2m\ell^2\sin^2\theta} - mg\ell\cos\theta + \frac{1}{2}m\ell^2\dot\theta^2, \quad L = m\ell v_0,\\
\frac{L^2}{2m\ell^2} &= \frac{L^2}{2m\ell^2\sin^2\theta_{\rm min}} - mg\ell\cos\theta_{\rm min},\\
(1-\cos^2\theta_{\rm min})\frac{L^2}{2m\ell^2} &= \frac{L^2}{2m\ell^2} - mg\ell\cos\theta_{\rm min}(1-\cos^2\theta_{\rm min}),\\
mg\ell\cos^2\theta_{\rm min} + \frac{L^2}{2m\ell^2}\cos\theta_{\rm min} - mg\ell &= 0,\\
\cos\theta_{\rm min} &= \frac{-L^2/(2m\ell^2) \pm \sqrt{L^4/(4m^2\ell^4) + 4m^2g^2\ell^2}}{2mg\ell}\\
\cos\theta_{\rm min} &= \frac{-mv_0^2/2 + \sqrt{m^2v_0^4/4 + 4m^2g^2\ell^2}}{2mg\ell}\\
&= \frac{-v_0^2 + \sqrt{v_0^4 + 16g^2\ell^2}}{4g\ell}.
\end{aligned}$$

b)

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}m\ell^2\sin^2\theta\dot\phi^2 + \frac{1}{2}m\ell^2\dot\theta^2 + mg\ell\cos\theta,\\
\frac{d}{dt}(m\ell^2\sin^2\theta\dot\phi) &= 0, \quad L = m\ell^2\sin^2\theta\dot\phi = \text{constant},\\
m\ell^2\ddot\theta &= -mg\ell\sin\theta + m\ell^2\dot\phi^2\sin\theta\cos\theta,\\
m\ell^2\ddot\theta &= -mg\ell\sin\theta + \frac{L^2\cos\theta}{m\ell^2\sin^3\theta},\\
\ddot\theta &= -\frac{g}{\ell}\sin\theta + \frac{L^2\cos\theta}{m^2\ell^4\sin^3\theta}.
\end{aligned}$$