your name______________________________________

Physics 321 Midterm #2a - Monday, March 19

FYI: For the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

the solutions are

$$\begin{aligned}
x &= A_1 e^{-\beta t}\cos\omega' t + A_2 e^{-\beta t}\sin\omega' t \quad \omega' = \sqrt{\omega_0^2 - \beta^2} \qquad \text{(under damped)} \\
x &= A e^{-\beta t} + B t e^{-\beta t}, \qquad \text{(critically damped)} \\
x &= A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}, \quad \beta_i = \beta \pm \sqrt{\beta^2 - \omega_0^2}, \quad \text{(over damped).}
\end{aligned}$$

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1. A particle of mass m moves in a harmonic oscillator potential,

$$V(x) = \frac{1}{2} m\omega_0^2 x^2,$$

experiences a drag force,

$$F_d = -2m\beta v,$$

and also experiences an external force

$$f = F_0 \Theta(t) e^{-\gamma t},$$

where $\Theta(t)$ is a step function.

(a) (10 pts) Solve for a particular solution of the form $x_p = Ce^{-\gamma t}$, such that it is a solution of the differential equation for $t > 0$.

(b) (15 pts) Find a solution that satisfies the initial conditions that $x(t = 0) = 0$ and $v(t = 0) = 0$.

(a)

$$\ddot{x} + 2\beta \dot{x} + \omega_0^2 x = F_0/m \, e^{-\gamma t}, \quad t > 0$$

$$x_p = Ce^{-\gamma t}$$

$$C\left[\gamma^2 - 2\beta\gamma + \omega_0^2\right] = F_0/m$$

$$C = \frac{(F_0/m)}{\gamma^2 - 2\beta\gamma + \omega_0^2}$$

(b)

$$x = A_1 e^{-\beta t} \cos\omega' t + A_2 e^{-\beta t} \sin\omega' t + Ce^{-\gamma t}$$

$$0 = A_1 + C$$

$$0 = -\beta A_1 + \omega' A_2 - \gamma C$$

$$A_1 = -C, \quad A_2 = \frac{-\beta C + \gamma C}{\omega'}$$

$$x = -Ce^{-\beta t}\cos\omega' t + \frac{\gamma C - \beta C}{\omega'} e^{-\beta t} \sin\omega t + Ce^{-\gamma t}$$

$$\omega' = \sqrt{\omega_0^2 - \beta^2}$$

your name________________________________

2. A particle of mass m moves in an undamped harmonic oscillator with potential

$$V(x) = \frac{1}{2}m\omega^2 x^2.$$

A periodic external force, where $F(t+\tau) = F(t)$ is applied,

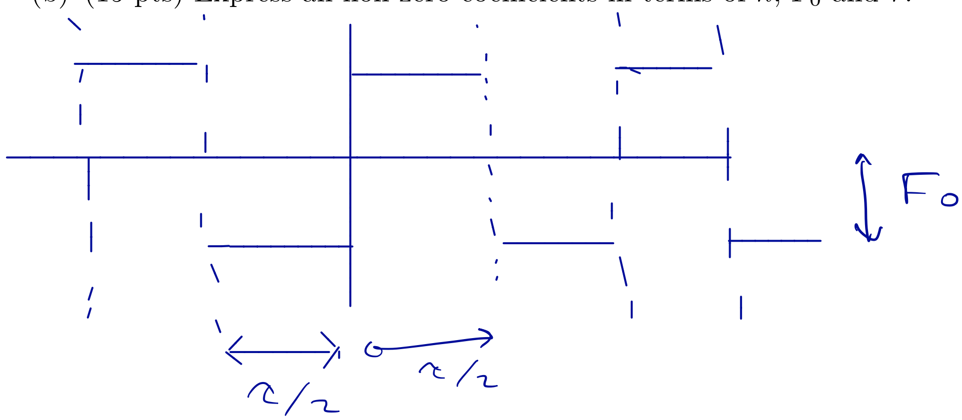
$$F(t) = \begin{cases} -F_0, & -\tau/2 < t < 0 \\ F_0, & 0 < t < \tau/2. \end{cases}$$

(a) (10 pts) Expanding the force in the form

$$F(t) = \sum_{n \geq 0} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t), \quad \omega_0 = 2\pi/\tau$$

list the values of n for which the coefficients $a_n \neq 0$, and the values of n for which $b_n \neq 0$.

(b) (15 pts) Express all non-zero coefficients in terms of n, F_0 and τ.



(a) $a_n = 0$ for all n

$b_n = 0$ for $n = 2, 4, 6, 8$

(b) $b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} F(t) \sin n\omega_0 t \, dt$, $\omega_0 = 2\pi/\tau$

$= \frac{4F_0}{\tau}\int_0^{\tau/2} dt \sin(n\omega_0 t)$

$= \frac{4F_0}{n\omega_0\tau}\left\{1 - \cos\frac{n\omega_0\tau}{2}\right\}$

$= \frac{4F_0}{n\omega_0\tau}\left[1 - (-1)^n\right] = \begin{cases} \frac{8F_0}{n\omega_0\tau}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$

$= \frac{4F_0}{n\pi}$ for $n = \text{odd}$

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Physics 321 Midterm #2b - Wednesday, March 21

3. A small asteroid of mass m is aimed at a heavy planet of mass M and radius R. The asteroid's kinetic energy is E_0 when the asteroid is far away.

   (a) (10 pts) Which of following quantities are conserved throughout the trajectory? Assume the initial impact parameter lies in the $\hat{y}$ direction and that the initial velocity is in the $\hat{x}$ direction, with the planet being located at the origin. Circle the ~~conserved quantities~~.

   quantities that remain constant throughout the trajectory.

   - The asteroid's kinetic energy
   - The asteroid's potential energy
   - The asteroid's total energy
   - The momentum component p_x
   - The momentum component p_y
   - The momentum component p_z
   - magnitude of the momentum $|\vec{p}|$
   - The radial velocity $v_r = \hat{r} \cdot \vec{v}$
   - The magnitude of the tangential velocity $v_t = |\hat{r} \times \vec{v}|$
   - The angular momentum component L_x
   - The angular momentum component L_y
   - The angular momentum component L_z
   - The magnitude of the angular momentum vector $|\vec{L}|$.

   (b) (15 pts) Solve for the cross section for a collision with the planet in terms of M, m, G and E_0.

$$L = m v_0 b, \quad V_0 = \sqrt{2E/m}$$

$$E = \frac{L^2}{2mR^2} - G\frac{Mm}{R}$$

$$b^2 = \frac{L^2}{m^2 v_0^2} = \frac{(L^2/2mR^2)}{m^2 v_0^2/(2mR^2)} = \frac{E + G\frac{Mm}{R}}{E} R^2$$

$$\pi b^2 = \sigma = \pi R^2 \left(1 + \frac{GMm}{RE}\right)$$

your name________________________________

4. A particle of mass m is in a circular orbit or radius R, moving according to a potential

$$V = -\frac{V_0}{r^{3/2}},$$

where $V_0 > 0$.

(a) (5 pts) What is the particle's speed while in a circular orbit? Give answer in terms of V_0, m and R.

(b) (5 pt) What is the particle's angular momentum, L, in the circular orbit? Give answer in terms of V_0, m and R.

(c) (5 pts) For a particle with this angular momentum, what is the effective radial potential? Provide a sketch. Label the radius of the circular orbit.

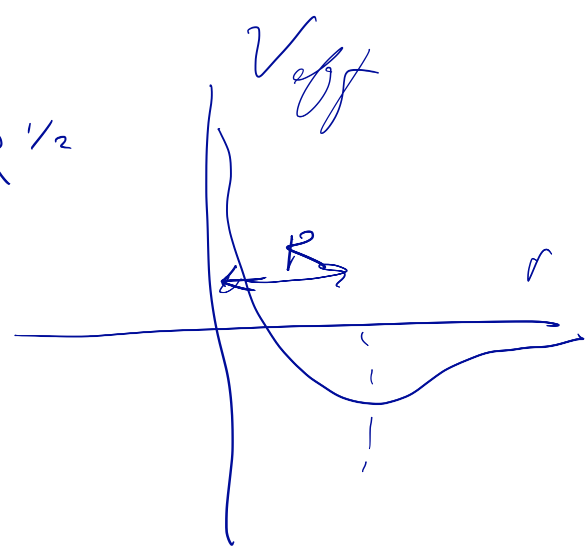
(d) (10 pts) What is the angular frequency of small oscillations of the radial distance r for such a particle with angular momentum L about the circular orbit? Give answer in terms of V_0, m and R.

(a) $$F/m = \frac{-(3/2)\,V_0/m}{R^{5/2}} = v^2/r$$

$$v = \frac{\left((3/2)(V_0/m)\right)^{1/2}}{R^{3/4}}$$

(b) $$L = m v R = \left(\frac{3}{2} m V_0\right)^{1/2} R^{1/4}$$

(c) $$V_{eff} = \frac{-V_0}{r^{3/2}} + \frac{\frac{3}{2} \cdot V_0 R^{1/2}}{2 \cdot r^2}$$



(d) $$k_{eff} = \left.\frac{d^2 V_{eff}}{dr^2}\right|_{r=R} = -\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\frac{V_0}{R^{7/2}} + \frac{9}{2}\frac{V_0}{R^{7/2}}$$

$$= \frac{3}{4}\frac{V_0}{R^{7/2}}$$

$$\omega = \sqrt{\frac{k_{eff}}{m}} = \left(\frac{3V_0}{4m}\right)^{1/2}\left(\frac{1}{R^{7/4}}\right)$$