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Physics 321 Midterm #2 - Wednesday, Nov. 19

FYI: For the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

the solutions are

$$\begin{aligned} x &= A_1 e^{-\beta t}\cos\omega' t + A_2 e^{-\beta t}\sin\omega' t \quad \omega' = \sqrt{\omega_0^2 - \beta^2} \quad \text{(under damped)} \\ x &= Ae^{-\beta t} + Bte^{-\beta t}, \quad \text{(critically damped)} \\ x &= A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}, \quad \beta_i = \beta \pm \sqrt{\beta^2 - \omega_0^2}, \quad \text{(over damped).} \end{aligned}$$

Coriolis and centrifugal forces

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}_{\rm real} - m\vec{\omega}\times\vec{\omega}\times\vec{r} - 2m\vec{\omega}\times\vec{v}.$$

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1. A small particle of mass m is aimed at a heavy target. The **REPULSIVE** Coulomb potential between the particles is

$$V(r) = \frac{\alpha}{r}.$$

   (a) (2 pts) If a collision occurs when the particles are separated by R or less, what is the minimum energy required for a collision?

   (b) (3 pts) If the impact parameter is b and the initial kinetic energy is E, what is the closest distance $r_{\rm min}$ reached during the trajectory?

   (c) (5 pts) Again assuming a collision occurs for if the particles come within R of one another, if the incoming energy of the particle is E what is the cross section σ for colliding with the target?

Solution:

a)

$$E_{\rm min} = \frac{\alpha}{R}.$$

b)

$$\begin{aligned} E &= \frac{L^2}{2mr_{\rm min}^2} + \frac{\alpha}{r_{\rm min}}, \\ L^2 &= 2mEb^2, \\ E &= \frac{Eb^2}{r_{\rm min}^2} + \frac{\alpha}{r_{\rm min}}, \\ Er_{\rm min}^2 - \alpha r_{\rm min} - Eb^2 &= 0, \\ r_{\rm min} &= \frac{\alpha + \sqrt{\alpha^2 + 4E^2b^2}}{2E}. \end{aligned}$$

c)

$$ER^2 - \alpha R - Eb^2 = 0,$$

$$\pi b^2 = \pi R^2\left(1 - \frac{\alpha}{RE}\right).$$

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2. (5 pts) A projectile is fired in Minneapolis (lattitude=45°) from a nearly horizontally aimed canon with a muzzle velocity v_0. If the cannon is initially aimed NORTH, and if the projectile travels a distance L, what is the deflection in the EAST-WEST direction due to the Coriolis force. Assume the deflection is much smaller than L, and specify whether the deflection is in the east or west direction. Refer to Earth's angular velocity as ω_0.

Solution:
Let $\hat{x}$ point east, $\hat{y}$ point north and $\hat{z}$ point up. Earth's rotational velocity is

$$\vec{\omega} = \omega_0 \frac{\hat{z} + \hat{y}}{\sqrt{2}}.$$

The Coriolis force in the x direction is then due to ω_z and the velocity, which is then the y direction.

$$F_x = 2m\frac{\omega_0}{\sqrt{2}}v_0$$

The displacement in the x direction is then

$$\begin{aligned}
\delta x &= \frac{1}{2}a_x t^2 = \frac{\sqrt{2}}{2}\omega_0 v_0 t^2, \\
t &= \frac{L}{v}, \\
\delta x &= \frac{\omega_0 L^2}{v_0\sqrt{2}}.
\end{aligned}$$

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3. After being dropped with zero initial velocity, a solid copper ball of mass m falls with a drag force $\gamma A v^2$, where A is the cross sectional area. The magnitude of the gravitational acceleration is g

   (a) (3 pts) Solve for the speed as a function of time.

   (b) (2 pts) If two solid copper balls A and B are dropped simultaneously, one with $R_B > R_A$, which ball falls more quickly? A or B? Explain your reasoning.

Solution:
a) Let down be the positive direction.

$$\begin{aligned} \frac{dv}{dt} &= g - (\gamma A/m)v^2, \\ t &= \int_0^v dv' \, \frac{1}{g - \gamma A f v'^2/m}, \\ &= \frac{v_0}{g} \int_0^{v/v_0} \frac{du}{1-u^2} \\ &= \frac{v_0}{g} \tanh^{-1}(v/v_0), \qquad v_0^2 \equiv mg/\gamma A \end{aligned}$$

$$v = v_0 \tanh(gt/v_0).$$

Note that the maximum velocity is v_0, which is the same velocity one would find if one solved for zero acceleration.

b) The ball with the higher $v_0^2 = mg/\gamma A$ will move more quickly. Since the mass goes as R^3 and the area as R^2, the larger ball will go more quickly. In other words, the gravitational force goes as R^3 and the drag as R^2.

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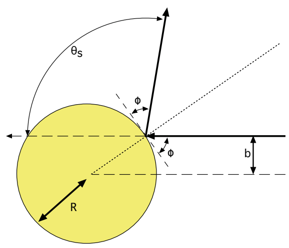

4. (5 pts extra credit - all or nothing). A point particle is fired at a spherical target of radius R. The particle bounces off the target elastically with scattering angle θ_s. The angle ϕ in the figure is only meant to show that the for a plane tangent to the surface, the angles relative to the surface are equal for the incoming and outgoing trajectories. Find the differential cross section $d\sigma/d\cos\theta_s$.

Solution:
An angle θ from the center of the target to the point of contact bisects the trajectory. The impact parameter is then

$$b = R\sin\theta,$$

and the scattering angle is

$$\theta_s = \pi - 2\theta.$$

Now, calculate the differential cross section

$$\begin{aligned} d\sigma &= 2\pi b db = 2\pi R^2 \sin\theta\cos\theta d\theta \\ &= \pi R^2 \sin 2\theta d\theta \\ &= \pi R^2 \sin(\pi - \theta_s) = \pi R^2 \sin\theta_s d\theta = \frac{\pi R^2}{2}\sin\theta_s d\theta_s \\ &= \frac{\pi R^2}{2} d\cos\theta_s, \\ \frac{d\sigma}{d\cos\theta_s} &= \frac{\pi R^2}{2}. \end{aligned}$$