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Physics 321 Exam #1 - Wednesday, Oct 8

FYI: For the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

the solutions are

$$\begin{aligned}
x &= A_1 e^{-\beta t}\cos\omega' t + A_2 e^{-\beta t}\sin\omega' t \quad \omega' = \sqrt{\omega_0^2 - \beta^2} \quad \text{(under damped)}\\
x &= A e^{-\beta t} + B t e^{-\beta t}, \quad \text{(critically damped)}\\
x &= A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}, \quad \beta_i = \beta \pm \sqrt{\beta^2 - \omega_0^2}, \quad \text{(over damped).}
\end{aligned}$$

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1. After being dropped with zero initial velocity, a solid copper ball of mass m falls with a drag force $\gamma A v^2$, where A is the cross sectional area. The magnitude of the gravitational acceleration is g

   (a) (3 pts) Solve for the speed as a function of time.

   (b) (2 pts) If two solid copper balls A and B are dropped simultaneously, one with $R_B > R_A$, which ball falls more quickly? A or B? Explain your reasoning.

Solution:
a) Let down be the positive direction.

$$\begin{aligned}
\frac{dv}{dt} &= g - (\gamma A/m)v^2,\\
t &= \int_0^v dv' \frac{1}{g - \gamma A v'^2/m},\\
&= \frac{v_0}{g}\int_0^{v/v_0} \frac{du}{1-u^2}\\
&= \frac{v_0}{g}\tanh^{-1}(v/v_0), \qquad v_0^2 \equiv mg/\gamma A
\end{aligned}$$

$$v = v_0 \tanh(gt/v_0).$$

Note that the maximum velocity is v_0, which is the same velocity one would find if one solved for zero acceleration.

b) The ball with the higher $v_0^2 = mg/\gamma A$ will move more quickly. Since the mass goes as R^3 and the area as R^2, the larger ball will go more quickly. In other words, the gravitational force goes as R^3 and the drag as R^2.

your name____________________________________

2. Ted and his iceboat have a combined mass of M_{T0}. Ted's boat slides without friction on top of a frozen lake. Ted's boat has a winch and he wishes to wind up a long heavy rope of mass M_{R0} and length L that is laid out in a straight line on the ice. Ted's boat starts at rest at one end of the rope, then brings the rope on board at a constant length per time of w. Clearly express all answers in term of M_{T0}, M_{R0}, L and w.

   (a) (1 pt) Before Ted turns on the winch, what is the position of the center of mass relative to the boat?

   (b) (1 pt) Immediately after the rope is entirely on board, what is Ted's displacement relative to his original position?

   (c) (1 pt) Immediately after the rope is entirely on board, where is the center of mass compared to Ted's original position?

   (d) (2 pts) Find Ted's velocity as a function of time

   (e) (1 pt) Find Ted's position as a function of time

Solution:

a)

$$X_{cm} = \frac{M_{R0}L/2}{M_{T0} + M_{R0}}.$$

b) Same as (a)

c) Same as (a)

d) Keeping momentum zero

$$\begin{aligned}
v_t - v_b &= w \\
(M_{T0} + wtM_{R0}/L)v_t &= -(M_{R0} - wtM_{R0}/L)v_b, \\
(M_{T0} + wtM_{R0}/L)v_t &= -(M_{R0} - wtM_{R0}/L)(v_t - w), \\
v_t &= w\frac{M_{R0} - wtM_{R0}/L}{M_{T0} + M_{R0}}
\end{aligned}$$

b)

$$\begin{aligned}
x_t &= \int v_t dt \\
&= \frac{M_{R0}}{M_{T0} + M_{R0}}wt - \frac{M_{R0}w^2}{2L(M_{T0} + M_{R0})}t^2.
\end{aligned}$$

your name____________________

3. Consider a **critically** damped one-dimensional harmonic oscillator. The particle has mass m, the spring constant is k and the drag force is $-bv$.

   (a) (2 pts) Write a general solution in terms of two arbitrary constants.

   (b) (3 pts) If at $t = 0$ the position is x_0 and the velocity is 0, find $x(t)$ for $t > 0$.

   (c) (4 pts) Assuming there is an external time-dependent force, $F_{\rm ext}(t) = f_0 \cos \omega t$, find the steady-state solution, i.e. the solution for large times.

Solution:
a)

$$x = Ae^{-\beta t} + Bte^{-\beta t}, \quad \beta = b/2m.$$

b) I.C. give

$$\begin{aligned} x_0 &= A, \quad -\beta A + B = 0, \\ A &= x_0, \quad B = \beta x_0, \\ x &= x_0 e^{-\beta t} + \beta x_0 t e^{-\beta t}. \end{aligned}$$

c) Find the particular solution assuming the form

$$x = Ce^{i\omega t},$$

and insert into the eq.s of motion,

$$\begin{aligned} \ddot{x} + 2\beta\dot{x} + \omega_0^2 x &= \frac{f_0}{m} e^{i\omega t}, \quad \beta \equiv b/2m, \omega_0^2 \equiv k/m, \\ (-\omega^2 + 2\beta i\omega + \omega_0^2)C &= \frac{f_0}{m}, \\ C &= \frac{f_0/m}{\omega_0^2 + 2\beta i\omega - \omega^2}, \\ \Re x &= \frac{(f_0/m)\cos(\omega t - \delta)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}, \\ \tan\delta &\equiv \frac{2\beta\omega}{\omega_0^2 - \omega^2}. \end{aligned}$$

Since the oscillator is critically damped ω_0 and β can be replaced with one another.